10.6 Oscillatory Stress, Dynamic Loading and Vibrations

Creep and relaxation experiments do not provide complete information concerning the mechanical behaviour of viscoelastic materials. These experiments usually provide test data in the time-range from 10 seconds to 10 years. It is often of interest to know the response of materials to loads of very short duration. For example, duration of the impact of a steel ball on a viscoelastic block may be of the order of 10^{-5} sec [1]. In order to be able to determine the response for such conditions, it is necessary to know the behaviour of a material at high rates of loading (or short duration loading).

The techniques and apparatus for investigating the response of a material to very short term loading are different to those involved in longer-term testing. For very short time loading it is more convenient to use oscillatory than static loading, and in order to predict the behaviour of a viscoelastic material subjected to an oscillatory load, one needs to formulate the theory based on oscillatory stresses and strains.

10.6.1 Oscillatory Stress

Consider a dynamic load of the form

$$\sigma(t) = \sigma_o \cos(\omega t) \tag{10.6.1}$$

where σ_o is the stress amplitude and ω is the angular frequency[2]. Assume that the resulting strain is of the form[3]

$$\varepsilon(t) = \varepsilon_o \cos(\omega t - \delta) \tag{10.6.2}$$

so that the strain is an oscillation at the same frequency as the stress but lags behind by a phase angle δ, Fig. 10.6.1. This angle is referred to as the **loss angle** of the material, for reasons which will become clear later.

Expanding the strain trigonometric terms,

$$\varepsilon(t) = \varepsilon_o \cos\delta \cos\omega t + \varepsilon_o \sin\delta \sin\omega t \tag{10.6.3}$$

The first term here is completely in phase with the input; the second term is completely out of phase with the input. If the phase angle δ is zero, then the stress and strain are in

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[1] dynamic experiments usually provide data from about 10^{-8} sec. to about 10^{3} sec. so there is a somewhat overlapping region where data can be obtained from both types of experiment

[2] when an oscillatory force is first applied, transient vibrations result at the natural frequency of the material – these soon die out leaving the vibrations at the source frequency

[3] if one substitutes 10.6.1 into the general constitutive equation 10.3.17, one sees that the strain and its derivatives contain sine and cosine terms, so that the strain must be of the general form $A\cos(\omega t) + B\sin(\omega t)$, where A and B are constants. For convenience, this can be written as $C\cos(\omega t - D)$ where C and D are new constants

phase (as happens with an ideal elastic material), whereas if $\delta = \pi / 2$, the stress and strain are completely out of phase.

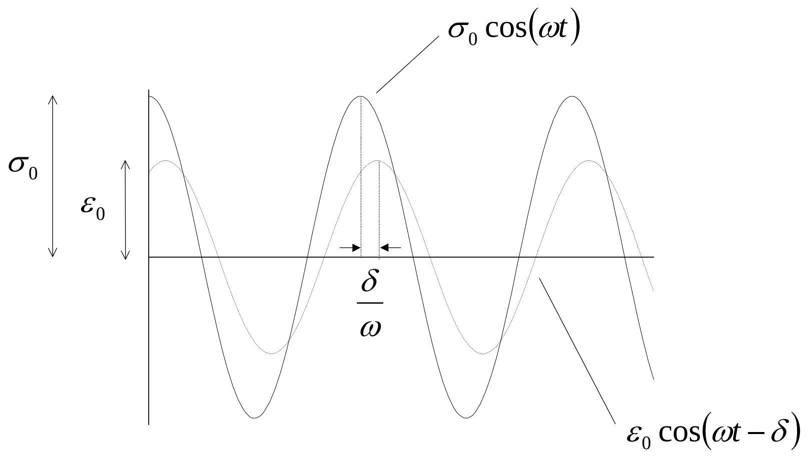

Figure 10.6.1: Oscillatory stress and strain

The Complex Compliance

Define

$$J_1 = \frac{\varepsilon_o}{\sigma_o} \cos \delta, \qquad J_2 = \frac{\varepsilon_o}{\sigma_o} \sin \delta \tag{10.6.4}$$

so that

$$\varepsilon(t) = \sigma_o \left(J_1 \cos \omega t + J_2 \sin \omega t \right) \tag{10.6.5}$$

The quantities J_1 and J_2 are a measure of how in, or out of, phase the stress is with the strain. The former, J_1, is called the **storage compliance** and the latter, J_2, is called the **loss compliance**. They are usually written as the components of a **complex compliance**, J^*:

$$J^* = J_1 - iJ_2 \tag{10.6.6}$$

If one has a stress input in the form of a sine function, then

$$\begin{aligned} \sigma(t) &= \sigma_o \sin(\omega t) \\ \varepsilon(t) &= \varepsilon_o \sin(\omega t - \delta) \\ &= \varepsilon_o \cos \delta \sin \omega t - \varepsilon_o \sin \delta \cos \omega t \\ &= \sigma_o \left(J_1 \sin \omega t - J_2 \cos \omega t \right) \end{aligned} \tag{10.6.7}$$

and again the storage compliance is a measure of the amount "in phase" and the loss compliance is a measure of the amount "out of phase".

The Complex Modulus

One can also regard of the strain as the input and the stress as the output. In that case one can write (δ is again the phase angle by which the strain lags behind the stress)

$$\begin{aligned}\varepsilon(t) &= \varepsilon_o \cos(\omega t) \\ \sigma(t) &= \sigma_o \cos(\omega t + \delta) \\ &= \sigma_o \cos\delta \cos\omega t - \sigma_o \sin\delta \sin\omega t\end{aligned} \tag{10.6.8}$$

This is in effect the same stress-strain relationship as that used above, only the stress/strain are shifted along the *t*-axis.

Define next the two new quantities

$$E_1 = \frac{\sigma_o}{\varepsilon_o}\cos\delta, \qquad E_2 = \frac{\sigma_o}{\varepsilon_o}\sin\delta \tag{10.6.9}$$

so that

$$\sigma(t) = \varepsilon_o\left(E_1 \cos\omega t - E_2 \sin\omega t\right) \tag{10.6.10}$$

Again, these quantities are a measure of how much the response is in phase with the input. The former, E_1, is called the **storage modulus** and the latter, E_2, is called the **loss modulus**. As with the compliances, they are usually written as the components of a **complex modulus**[4], E^*:

$$E^* = E_1 + iE_2 \tag{10.6.11}$$

Again, if one has a sinusoidal strain as input, one can write

$$\begin{aligned}\varepsilon(t) &= \varepsilon_o \sin(\omega t) \\ \sigma(t) &= \sigma_o \sin(\omega t + \delta) \\ &= \sigma_o \cos\delta \sin\omega t + \sigma_o \sin\delta \cos\omega t \\ &= \varepsilon_o\left(E_1 \sin\omega t + E_2 \cos\omega t\right)\end{aligned} \tag{10.6.12}$$

It is apparent from the above that

$$J^* E^* = 1 \tag{10.6.13}$$

which is a much simpler relationship than that between the creep compliance function and the relaxation modulus (which involved Laplace transforms, Eqn. 10.5.28).

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[4] typical values for the storage and loss moduli for a polymer would be around $E_1 = 10\ \text{MPa}$, $E_2 = 0.1\ \text{MPa}$. The ratio of the amplitudes is called the **dynamic modulus**, $\left|E^*\right| = \sigma_o / \varepsilon_o$.

Complex Formulation

The above equations can be succinctly written using a complex formulation, using Euler's formula

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta \tag{10.6.14}$$

For a stress input,

$$\begin{aligned} \sigma(t) &= \sigma_o e^{i\omega t} \\ \varepsilon(t) &= \varepsilon_o e^{i(\omega t-\delta)} \\ &= \varepsilon_o \left(\cos\delta - i\sin\delta\right) e^{i\omega t} \\ &= \sigma_o \left[J_1 - iJ_2\right] e^{i\omega t} \\ &= \sigma_o J^* e^{i\omega t} \end{aligned} \tag{10.6.15}$$

The creep compliance function $J(t)$ is the strain response to a unit load. In the same way, from 10.6.15, the complex compliance J^* can be interpreted as the strain amplitude response to a sinusoidal stress input of unit magnitude.

Similarly, for a strain input, one has

$$\begin{aligned} \varepsilon(t) &= \varepsilon_o e^{i\omega t} \\ \sigma(t) &= \sigma_o e^{i(\omega t+\delta)} \\ &= \sigma_o \left(\cos\delta + i\sin\delta\right) e^{i\omega t} \\ &= \varepsilon_o E^* e^{i\omega t} \end{aligned} \tag{10.6.16}$$

and the term in brackets is, by definition, the complex modulus E^*.

The relationship between the complex compliance/modulus and the differential constitutive equation

Putting $\sigma(t) = \sigma_o e^{i\omega t}$ and the resulting strain $\varepsilon(t) = \varepsilon_o e^{(i\omega-\delta)t}$ into the general differential operator form of the constitutive equation 10.3.19, one has

$$\begin{aligned} &\left[p_o + p_1(i\omega) + p_2(i\omega)^2 + p_3(i\omega)^3 + \cdots\right]\sigma_o e^{i\omega t} \\ &\qquad = \left[q_o + q_1(i\omega) + q_2(i\omega)^2 + q_3(i\omega)^3 + \cdots\right]\sigma_o J^* e^{i\omega t} \end{aligned} \tag{10.6.17}$$

This equation thus gives the relationship between the complex compliance and the constants p_i, q_i. A similar relationship can be found for the complex modulus:

$$E^* = \frac{q_o + q_1(i\omega) + q_2(i\omega)^2 + q_3(i\omega)^3 + \cdots}{p_o + p_1(i\omega) + p_2(i\omega)^2 + p_3(i\omega)^3 + \cdots} \tag{10.6.18}$$

Again one sees that $J^* E^* = 1$.

From 10.6.17-18, the complex compliance and complex modulus are functions of the frequency ω, and thus, from the definitions 10.6.4, 10.6.6, 10.6.9, 10.6.11, so is the phase angle δ. Thus ω is the primary variable influencing the viscoelastic properties (whereas time *t* was used for this purpose in the analysis of static loading).

The relationship between the complex compliance/modulus and the creep compliance/ relaxation modulus

It can be shown[5] that the complex compliance $J^*(\omega)$ and the complex modulus $E^*(\omega)$ are related to the creep compliance $J(t)$ and relaxation modulus $E(t)$ through

$$\begin{aligned} J^*(\omega) &= (i\omega)L\left[J(t)\right]_{s=i\omega} \\ E^*(\omega) &= (i\omega)L\left[E(t)\right]_{s=i\omega} \end{aligned}, \qquad (10.6.19)$$

Here, the Laplace transform is first taken and then evaluated at $s = i\omega$ [6].

A Note on Frequency

Frequencies below 0.1 Hz are associated with seismic waves. Vibrations of structures and solid objects occur from about 0.1 Hz to 10 kHz depending on the size of the structure. Stress waves from 20 Hz to 20 kHz are perceived as sound - above 20 kHz is the ultrasonic range. Frequencies above 10^{12} Hz correspond to molecular vibration and represent an upper limit for stress waves in real solids.

10.6.2 Example: The Maxwell Model

The constitutive equation for the Maxwell model is given by Eqn. 10.3.6,

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon} \qquad (10.6.20)$$

Consider an oscillatory stress $\sigma = \sigma_o \cos(\omega t)$. We thus have[7]

$$\begin{aligned} \sigma_o \cos(\omega t) - \frac{\eta}{E}\omega\sigma_o \sin(\omega t) = \eta\dot{\varepsilon} &\to \int d\varepsilon = \sigma_o \left\{ -\frac{\omega}{E}\int \sin(\omega t)dt + \frac{1}{\eta}\int \cos(\omega t)dt \right\} \\ &\to \varepsilon(t) = \sigma_o \left\{ \frac{1}{E}\cos(\omega t) + \frac{1}{\omega\eta}\sin(\omega t) \right\} \end{aligned} \qquad (10.6.21)$$

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[5] using Fourier transform theory for example

[6] J_1 and J_2 are also related to each other (as are E_1 and E_2) by an even more complicated rule known as the **Kramers-Kronig relation**

[7] the constant of integration is zero (assuming that the initial strain is that in the spring, σ_o / E).

Thus the complex compliance is

$$J^* = J_1 - iJ_2 = \frac{1}{E} - i\frac{1}{\omega\eta} \tag{10.6.22}$$

This result can be obtained more easily using the relationship between the complex compliance and the constitutive equation: the constitutive equation can be rewritten as

$$p_o\sigma + p_1\dot{\sigma} = q_0\varepsilon + q_1\dot{\varepsilon}, \quad \text{where} \quad p_o = 1,\ p_1 = \frac{\eta}{E},\ q_0 = 0,\ q_1 = \eta \tag{10.6.23}$$

From Eqn. 10.6.17,

$$J^* = \frac{p_o + p_1(i\omega) + p_2(i\omega)^2 + \cdots}{q_o + q_1(i\omega) + q_2(i\omega)^2 + \cdots} = \frac{1 + (\eta/E)(i\omega)}{\eta(i\omega)} = \frac{1}{E} - i\frac{1}{\omega\eta} \tag{10.6.24}$$

Also, the complex modulus is related to the complex compliance through 10.6.13, $E^* = 1/J^*$, so that

$$E^* = \frac{(\omega\eta)^2 E}{(\omega\eta)^2 + E^2} + i\frac{\omega\eta E^2}{(\omega\eta)^2 + E^2} \tag{10.6.25}$$

For very low frequencies, $\omega \to 0$, $\sin(\omega t)/\omega \to t$, and the response, as expected, reduces to that for a static load, $\varepsilon(t) = \sigma_o\left(1/E + t/\eta\right)$.

For very high frequencies, $1/\omega \to 0$, and the response is $\varepsilon(t) = \left(\sigma_o / E\right)\cos(\omega t)$. Thus the strain is completely in-phase with the load, but the dash-pot is not moving – it has no time to respond at such high frequencies - the spring/dash-pot model is reacting like an isolated spring, that is, like a solid, with no fluid behaviour.

10.6.3 Energy Dissipation

Because the equations 10.6.12

$$\varepsilon(t) = \varepsilon_o \sin(\omega t), \quad \sigma(t) = \sigma_o \sin(\omega t + \delta) \tag{10.6.26}$$

are the parametric equations for an ellipse, that is, they trace out an ellipse for values of *t*, the stress-strain curve for an oscillatory stress is an elliptic hysteresis loop, Fig. 10.6.2.

The work done in stressing a material (per unit volume) is given by

$$W = \int \sigma d\varepsilon \tag{10.6.27}$$

The energy lost ΔW through internal friction and heat is given by the area of the ellipse. Thus

$$\Delta W = \int_{t_1}^{t_1+T} \sigma d\varepsilon = \int_{t_1}^{t_1+T} \sigma \frac{d\varepsilon}{dt} dt \tag{10.6.28}$$

where t_1 is some starting time and T is the period of oscillation, $T = 2\pi / \omega$. Substituting in Eqns. 10.6.26 for strain and stress then gives

$$\begin{aligned}\Delta W &= \omega\sigma_o\varepsilon_o \int_{t_1}^{t_1+T} \sin(\omega t + \delta)\cos(\omega t) dt \\ &= \frac{1}{2}\omega\sigma_o\varepsilon_o \int_{t_1}^{t_1+T} [\sin(2\omega t + \delta) + \sin\delta] dt \\ &= \frac{1}{2}\omega\sigma_o\varepsilon_o \left[-\frac{\cos(2\omega t + \delta)}{2\omega} + t\sin\delta \right]_{t_1}^{t_1+T}\end{aligned} \tag{10.6.29}$$

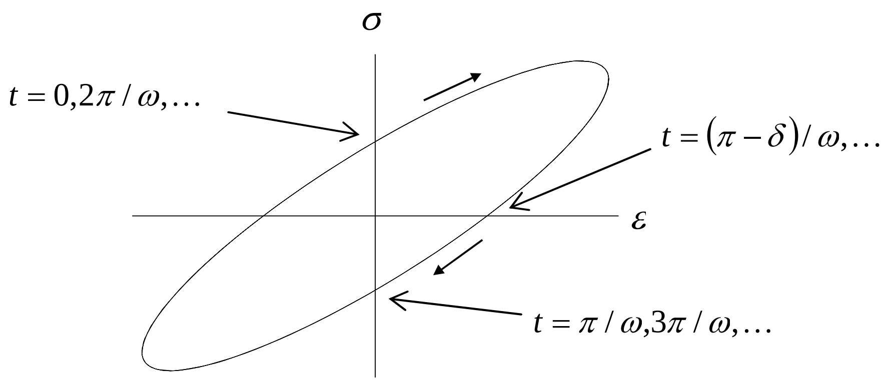

Figure10.6.2: Elliptic Stress-Strain Hysteresis Loop

Taking $t_1 = 0$ then gives[8]

$$\boxed{\Delta W = \pi\sigma_o\varepsilon_o \sin\delta} \qquad \textbf{Energy Loss} \tag{10.6.30}$$

When $\delta = 0$, the energy dissipated is zero, as in an elastic material. It can also be seen that

$$\Delta W = \pi\varepsilon_0^2 E_2 = \pi\sigma_0^2 J_2 \tag{10.6.31}$$

and hence the names *loss* modulus and *loss* compliance.

[8] the same result is obtained for $\sigma = \sigma_o \sin(\omega t),\ \varepsilon = \varepsilon_o \sin(\omega t - \delta)$ or when the stress and strain are cosine functions

Damping Energy

The energy stored after one complete cycle is zero since the material has returned to its original configuration. The maximum energy stored during any one cycle can be computed by integrating the increment of work $\sigma d\varepsilon$ from zero up to a maximum stress, that is over one quarter the period T of one cycle. Thus, integrating from $t_1 = -\delta/\omega$ (where $\sigma = 0$) to $t_2 = t_1 + \pi/2\omega$, Fig. 10.6.3[9]

$$W = \sigma_o \varepsilon_o \left[\frac{\cos\delta}{2} + \frac{\pi}{4}\sin\delta \right] \tag{10.6.32}$$

The second term is $\pi\sigma_o\varepsilon_o \sin\delta/4$, which is one quarter of the energy dissipated per cycle, and so can be considered to represent the dissipated energy. The remaining, first, term represents the area of the shaded triangle in Fig. 10.6.3 and can be considered to be the energy stored, $W_s = \sigma_o\varepsilon_o \cos\delta/2$ (it reduces to the elastic solution $W = \sigma_o\varepsilon_o/2$ when $\delta = 0$).

The **damping energy** of a viscoelastic material is defined as $\Delta W/W_S$, where W_S is the maximum energy the system can store in a given stress/strain amplitude. Thus (dividing ΔW by 4 so it is consistent with the integration over a quarter-cycle to obtain the stored energy)

$$\boxed{\frac{\Delta W}{W_s} = \frac{\pi}{2}\tan\delta} \qquad \textbf{Damping Energy} \tag{10.6.33}$$

Thus the damping ability of a linearly viscoelastic material is only dependent on the phase/loss angle δ.

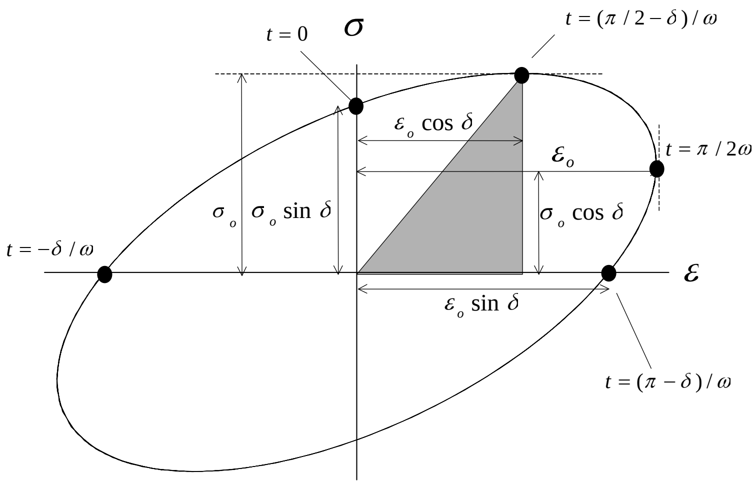

Figure10.6.3: Elliptic Stress-Strain Hysteresis Loop

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[9] or one could integrate from zero to maximum strain, over $[0, \pi/2\omega]$, giving the same result

The quantity $\tan\delta$ is known as the **mechanical loss**, or the **loss tangent**. It can be considered to be the fundamental measure of damping in a linear material (other measures, for example δ, $2\pi\tan\delta$, etc., are often used)[10]. Typical values for a range of materials at various temperatures and frequencies are shown in Table 10.6.1.

Material	Temperature	Frequency (ν)	Loss Tangent ($\tan\delta$)
Sapphire	4.2 K	30 kHz	2.5×10^{-10}
Sapphire	rt	30 kHz	5×10^{-9}
Silicon	rt	20 kHz	3×10^{-8}
Quartz	rt	1 MHz	$\approx10^{-7}$
Aluminium	rt	20 kHz	$<10^{-5}$
Cu-31%Zn	rt	6 kHz	9×10^{-5}
Steel	rt	1 Hz	0.0005
Aluminium	rt	1 Hz	0.001
Fe-0.6%V	33°C	0.95 Hz	0.0016
Basalt	rt	0.001-0.5 Hz	0.0017
Granite	rt	0.001-0.5 Hz	0.0031
Glass	rt	1 Hz	0.0043
Wood	rt	≈ 1 Hz	0.02
Bone	37°C	1-100 Hz	0.01
Lead	rt	1-15 Hz	0.029
PMMA	rt	1 Hz	0.1

Table 10.6.1: Loss Tangents of Common Materials[11]

10.6.4 Impact

Consider the impact of a viscoelastic ball dropped from a height h_d onto a rigid floor. During the impact, a proportion of the initial potential energy mgh_d, which is now kinetic energy $\frac{1}{2}mv^2$, where v is the velocity at impact, is lost and only some is stored. The stored energy is converted back to kinetic energy which drives the ball up on the rebound, reaching a height $h_r < h_d$, with final potential energy mgh_r. The ratio of the two heights is[12]

$$f \equiv \frac{h_r}{h_d} = \frac{mgh_r}{mgh_d} = \frac{W_s}{W_s + W_d} \tag{10.6.34}$$

where W_s is the energy stored and W_d is the energy dissipated during the impact.

[10] some investigators recommend that one uses the maximum storable energy when $\delta = 0$, in which case the stored energy is $\sigma_o\varepsilon_o / 2$ and the damping measure would be $\Delta W / W_S = \pi\sin\delta / 2$

[11] from Table 7.1 of Viscoelastic Solids, by R. S. Lakes, CRC Press, 1999

[12] the **coefficient of restitution** e is defined as the ratio of the velocities before and after impact, $e = v_r / v_d$, so $f = e^2$.

The impact event can be approximated by a half-cycle of the oscillatory stress-strain curve, Fig. 10.6.4. Integrating over $[0, \pi/\omega]$ or $[-\delta/\omega, (\pi-\delta)/\omega]$, one has[13]

$$W = \frac{1}{2}\sigma_o \varepsilon_o [\cos\delta + \pi \sin\delta] \tag{10.6.35}$$

and so the "height lost" is given by

$$f = 1 - \frac{W_d}{W_s + W_d} \approx 1 - \frac{W_d}{W_s} = 1 - \pi \tan\delta \tag{10.6.36}$$

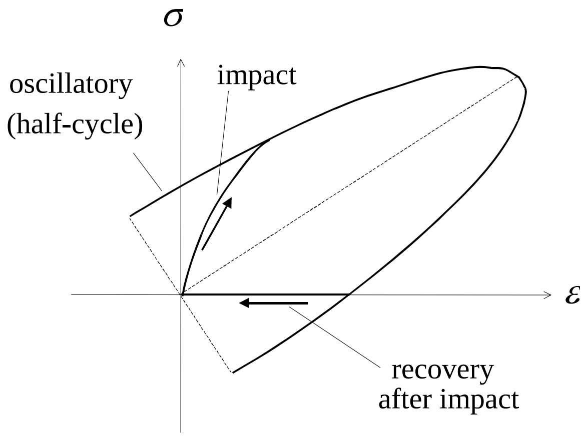

Figure10.6.4: Impact approximated as a half-cycle of oscillatory stress and strain

Note some other approximations made:
(i) energy losses due to air resistance, friction and radiation of sound energy during impact have been neglected
(ii) in a real impact, the stress and strain are both initially zero. In the current analysis, when one of these quantities is zero, the other is finite, and this will inevitably introduce some error[14].

10.6.5 Damping of Vibrations

The inertial force in many applications can be neglected. However, when dealing with vibrations, the product of acceleration times mass can be appreciable when compared to the other forces present.

Vibrational damping can be examined by looking at a simple oscillator with one degree of freedom, Fig. 10.6.5. A mass m is connected to a wall by a viscoelastic bar of length L and cross sectional area A. The motion of the system is described by the equations

[13] although it might be more accurate to integrate over $[0, (\pi-\delta)/\omega]$

[14] as mentioned, there is a transient term involved in the oscillation which has been ignored, and which dies out over time, leaving the strain to lag behind the stress at a constant phase angle

Dynamic equation: $m\ddot{x} + F = 0$
Kinematic relation: $\varepsilon = x / L$
Constitutive relation: (depends on model)

Assuming an oscillatory motion, $x = x_o e^{i\omega t}$, and using the first two of these,

$$-\omega^2 m x_o e^{i\omega t} + A\sigma = 0 \qquad \to \quad \sigma = \frac{x_o}{L}\frac{L\omega^2 m}{A}e^{i\omega t} = \varepsilon_o \left[\frac{L\omega^2 m}{A}\right]e^{i\omega t} \tag{10.6.37}$$

The quantity in brackets is the complex modulus E^* (see Eqn. 10.6.16).

As an example, for the Maxwell model (see Eqn. 10.6.24)

$$E^* = \left[\frac{1}{E} - \frac{i}{\eta\omega}\right]^{-1} \tag{10.6.38}$$

and so

$$\frac{1}{E}\omega^2 - \frac{i}{\eta}\omega = \frac{A}{Lm}, \tag{10.6.39}$$

which can be solved to get

$$\omega = \left\{\frac{E}{2\eta}i \pm \sqrt{\frac{AE}{Lm} - \frac{E^2}{4\eta^2}}\right\} \tag{10.6.40}$$

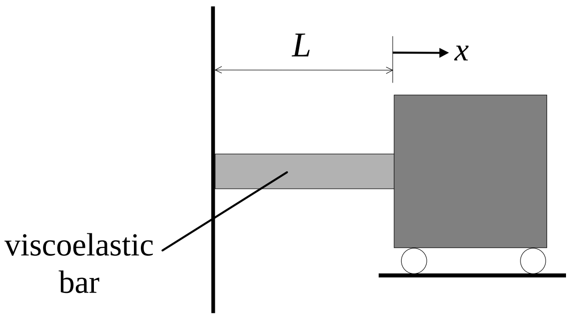

Figure10.6.5: Vibration

If m is small or E is large (and E/η is not too large) the root has a real part, v say, so that

$$\omega = i(E/2\eta) \pm v \tag{10.6.41}$$

and one has the damped vibration

$$\begin{aligned} x &= x_o\left(c_1 e^{i\omega_1 t} + c_2 e^{i\omega_2 t}\right) \\ &= x_o e^{-(E/2\eta)t}\left(c_1 e^{ivt} + c_2 e^{-ivt}\right) \\ &= x_o e^{-(E/2\eta)t}\left(A\cos(vt) + B\sin(vt)\right) \end{aligned} \qquad (10.6.42)$$

If, on the other hand, the mass is large or the spring compliant, one gets a pure imaginary root, $\omega = i(E/2\eta) \pm iv$, so that $i\omega$ is real (and less than zero) and one has the aperiodic damping

$$x = x_o\left(c_1 e^{(E/2\eta+v)t} + c_2 e^{(E/2\eta-v)t}\right) \qquad (10.6.43)$$

10.6.6 Problems

1. Use the differential form of the constitutive equation for a linearly viscoelastic material to derive the *complex compliance*, the *complex modulus*, and the *loss tangent* for a Kelvin material. (put the first two in the form $\alpha + i\beta$). Use your expression for the complex compliance to derive the strain response to a stress $\sigma_o \cos(\omega t)$, in terms of $\sigma_o, \omega, t, E, \eta$, in the form
$$\varepsilon(t) = \sigma_o\left(A\cos\omega t + B\sin\omega t\right)$$
What happens at very low frequencies?