7.4 The Elementary Beam Theory

In this section, problems involving long and slender beams are addressed. As with pressure vessels, the geometry of the beam, and the specific type of loading which will be considered, allows for approximations to be made to the full three-dimensional linear elastic stress-strain relations.

The beam theory is used in the design and analysis of a wide range of structures, from buildings to bridges to the load-bearing bones of the human body.

7.4.1   The Beam

The term **beam** has a very specific meaning in engineering mechanics: it is a component that is designed to support **transverse loads**, that is, loads that act perpendicular to the longitudinal axis of the beam, Fig. 7.4.1. The beam supports the load by *bending only*. Other mechanisms, for example twisting of the beam, are not allowed for in this theory.
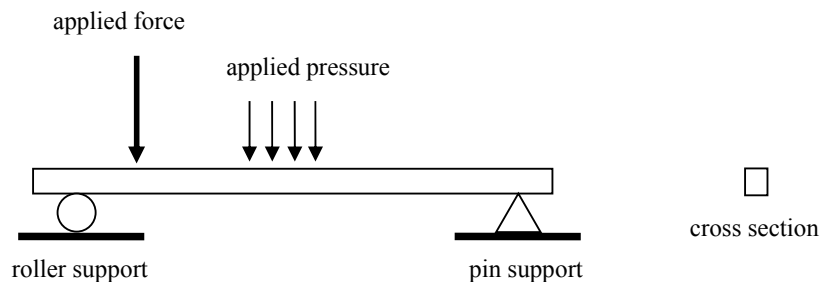

applied force

applied pressure

cross section

roller support

pin support

Figure 7.4.1: A supported beam loaded by a force and a distribution of pressure

It is convenient to show a two-dimensional cross-section of the three-dimensional beam together with the beam cross section, as in Fig. 7.4.1. The beam can be supported in various ways, for example by roller supports or pin supports (see section 2.3.3). The cross section of this beam happens to be rectangular but it can be any of many possible shapes.

It will assumed that the beam has a **longitudinal plane of symmetry**, with the cross section symmetric about this plane, as shown in Fig. 7.4.2. Further, it will be assumed that the loading and supports are also symmetric about this plane. With these conditions, the beam has no tendency to twist and will undergo bending only[1].
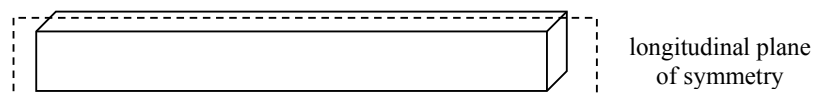

longitudinal plane
of symmetry

Figure 7.4.2: The longitudinal plane of symmetry of a beam

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[1] certain very special cases, where there is *not* a plane of symmetry for geometry and/or loading, can lead also to bending with no twist, but these are not considered here

Imagine now that the beam consists of many fibres aligned longitudinally, as in Fig. 7.4.3. When the beam is bent by the action of downward transverse loads, the fibres near the top of the beam contract in length whereas the fibres near the bottom of the beam extend. Somewhere in between, there will be a plane where the fibres do not change length. This is called the **neutral surface**. The intersection of the longitudinal plane of symmetry and the neutral surface is called the **axis of the beam**, and the deformed axis is called the **deflection curve**.
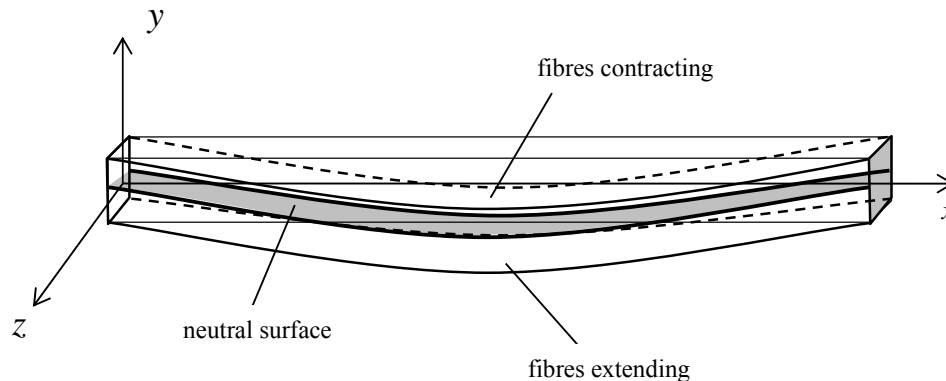

Figure 7.4.3: the neutral surface of a beam

A conventional coordinate system is attached to the beam in Fig. 7.4.3. The x axis coincides with the (longitudinal) axis of the beam, the y axis is in the transverse direction and the longitudinal plane of symmetry is in the $x - y$ plane, also called the **plane of bending**.

7.4.2   Moments and Forces in a Beam

Normal and shear stresses act over any cross section of a beam, as shown in Fig. 7.4.4. The normal and shear stresses acting on each side of the cross section are equal and opposite for equilibrium, Fig. 7.4.4b. The normal stresses σ will vary over a section during bending. Referring again to Fig. 7.4.3, over one part of the section the stress will be tensile, leading to extension of material fibres, whereas over the other part the stresses will be compressive, leading to contraction of material fibres. This distribution of normal stress results in a moment M acting on the section, as illustrated in Fig. 7.4.4c. Similarly, shear stresses τ act over a section and these result in a shear force V.

The beams of Fig. 7.4.3 and Fig. 7.4.4 show the normal stress and deflection one would expect when a beam bends downward. There are situations when parts of a beam bend upwards, and in these cases the signs of the normal stresses will be opposite to those shown in Fig. 7.4.4. However, the moments (and shear forces) shown in Fig. 7.4.4 will be regarded as *positive*. This sign convention to be used is shown in Fig. 7.4.5.
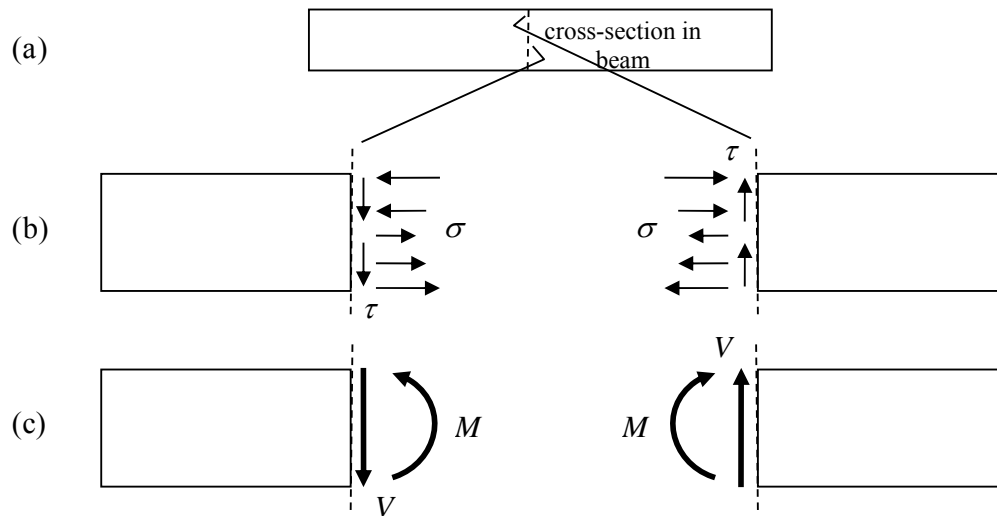
Figure 7.4.4: stresses and moments acting over a cross-section of a beam; (a) a cross-section, (b) normal and shear stresses acting over the cross-section, (c) the moment and shear force resultant of the normal and shear stresses
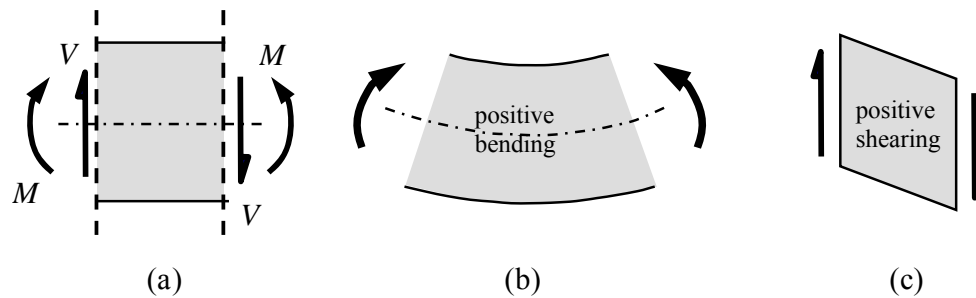

Figure 7.4.5: sign convention for moments and shear forces

Note that the sign convention for the shear stress conventionally used the beam theory conflicts with the sign convention for shear stress used in the rest of mechanics, introduced in Chapter 3. This is shown in Fig. 7.4.6.
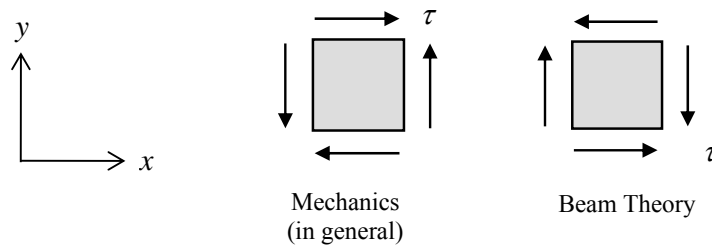

Figure 7.4.6: sign convention for shear stress in beam theory

The moments and forces acting within a beam can in many simple problems be evaluated from equilibrium considerations alone. Some examples are given next.

Example 1

Consider the **simply supported** beam in Fig. 7.4.7. From the loading, one would expect the beam to deflect something like as indicated by the deflection curve drawn. The reaction at the roller support, end A, and the vertical reaction at the pin support[2], end B, can be evaluated from the equations of equilibrium, Eqns. 2.3.3:

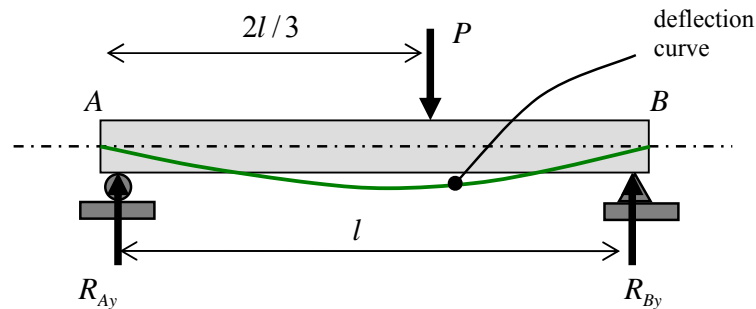$$R_{Ay} = P/3, \quad R_{By} = 2P/3 \tag{7.4.1}$$

Figure 7.4.7: a simply supported beam

The moments and forces acting *within* the beam can be evaluated by taking free-body diagrams of sections of the beam. There are clearly two distinct regions in this beam, to the left and right of the load. Fig. 7.4.8a shows an arbitrary portion of beam representing the left-hand side. A coordinate system has been introduced, with x measured from A.[3] An unknown moment M and shear force V act at the end. A *positive* moment and force have been drawn in Fig. 7.4.8a. From the equilibrium equations, one finds that the shear force is constant but that the moment varies linearly along the beam:

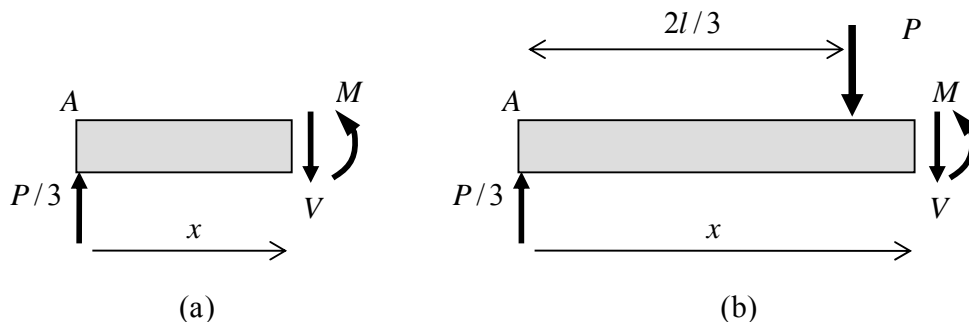$$V = \frac{P}{3}, \qquad M = \frac{P}{3}x \qquad \left(0 < x < \frac{2l}{3} \right) \tag{7.4.2}$$

(a)   (b)

Figure 7.4.8: free body diagrams of sections of a beam

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[2] the horizontal reaction at the pin is zero since there are no applied forces in this direction; the beam theory does not consider such types of (axial) load; further, one does not have a pin at each support, since this would prevent movement in the horizontal direction which in turn would give rise to forces in the horizontal direction – hence the pin at one end and the roller support at the other end

[3] the coordinate x can be measured from any point in the beam; in this example it is convenient to measure it from point A

Cutting the beam to the right of the load, Fig. 7.4.8b, leads to

$$V = -\frac{2P}{3}, \qquad M = \frac{2P}{3}(l-x) \qquad (\frac{2l}{3} < x < l) \qquad (7.4.3)$$

The shear force is negative, so acts in the direction opposite to that initially assumed in Fig. 7.4.8b.

The results of the analysis can be displayed in what are known as a **shear force diagram** and a **bending moment diagram**, Fig. 7.4.9. Note that there is a "jump" in the shear force at $x = 2l/3$ equal to the applied force, and in this example the bending moment is everywhere positive.
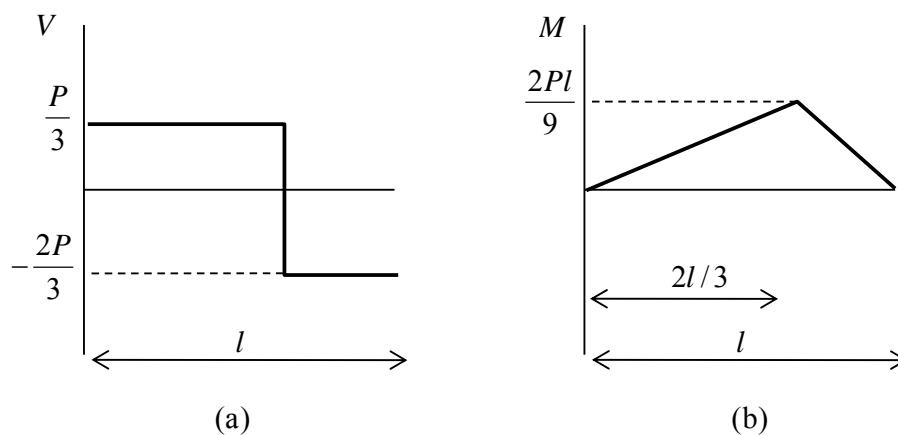

(a)   (b)

Figure 7.4.9: results of analysis; (a) shear force diagram, (b) bending moment diagram

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Example 2

Fig. 7.4.10 shows a **cantilever**, that is, a beam supported by clamping one end (refer to Fig. 2.3.8). The cantilever is loaded by a force at its mid-point and a (negative) moment at its end.
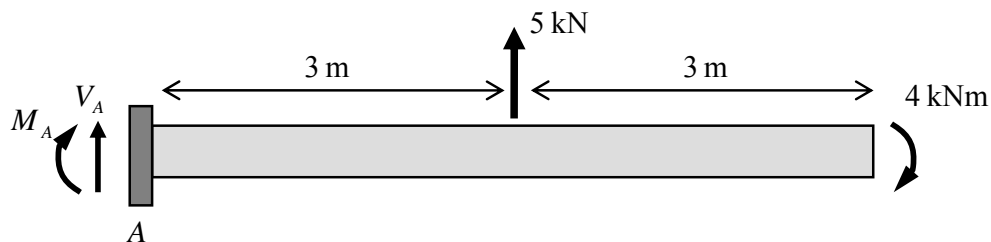

Figure 7.4.10: a cantilevered beam loaded by a force and moment

Again, positive unknown reactions M_A and V_A are considered at the support A. From the equilibrium equations, one finds that

$$M_A = 11 \text{ kNm}, \quad V_A = -5 \text{ kN} \tag{7.4.4}$$

As in the previous example, there are two distinct regions along the beam, to the left and to the right of the applied concentrated force. Again, a coordinate x is introduced and the beam is sectioned as in Fig. 7.4.11. The unknown moment M and shear force V can then be evaluated from the equilibrium equations:

$$
\begin{aligned}
V &= -5 \text{ kN}, & M &= 11 - 5x \text{ kNm} & (0 < x < 3) \\
V &= 0, & M &= -4 \text{ kNm} & (3 < x < 6)
\end{aligned}
\tag{7.4.5}
$$

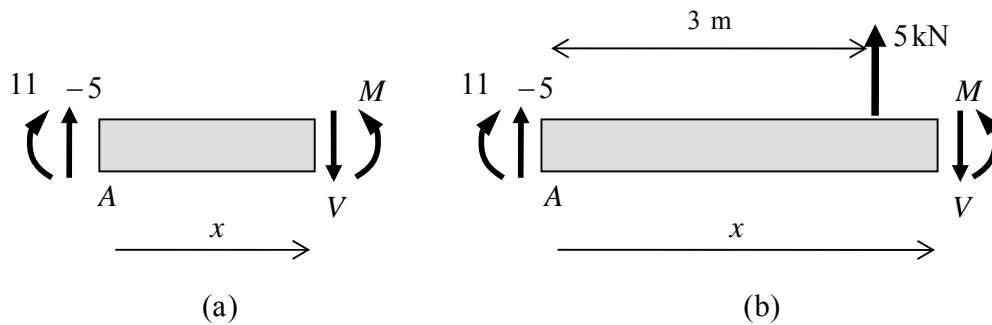(a)   (b)

Figure 7.4.11: free body diagrams of sections of a beam

The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.12.

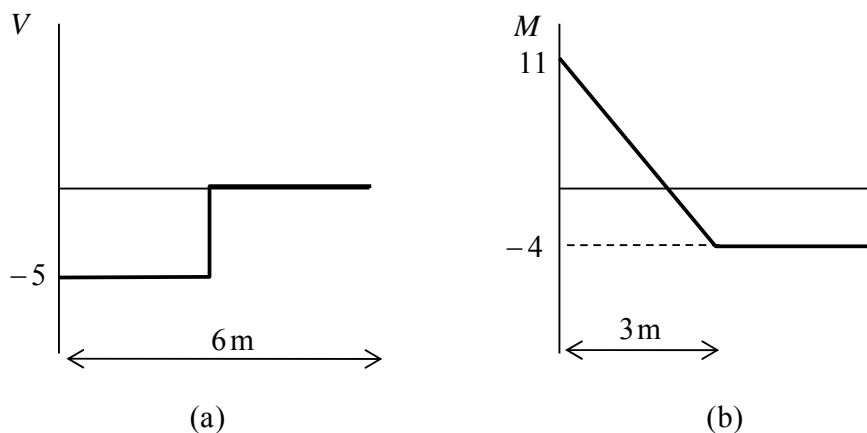(a)   (b)

Figure 7.4.12: results of analysis; (a) shear force diagram, (b) bending moment diagram

In this example the beam experiences negative bending moment over most of its length.

■

Example 3

Fig. 7.4.13 shows a simply supported beam subjected to a distributed load (force per unit length). The load is uniformly distributed over half the length of the beam, with a triangular distribution over the remainder.
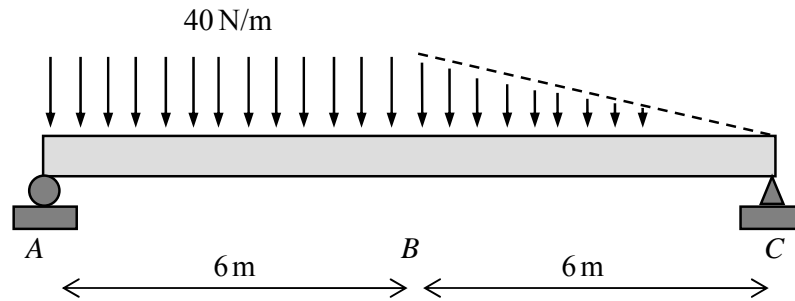

40 N/m

A   6 m   B   6 m   C

Figure 7.4.13: a beam subjected to a distributed load

The unknown reactions can be determined by replacing the distributed load with statically equivalent forces as in Fig. 7.4.14 (see §3.1.2). The equilibrium equations then give

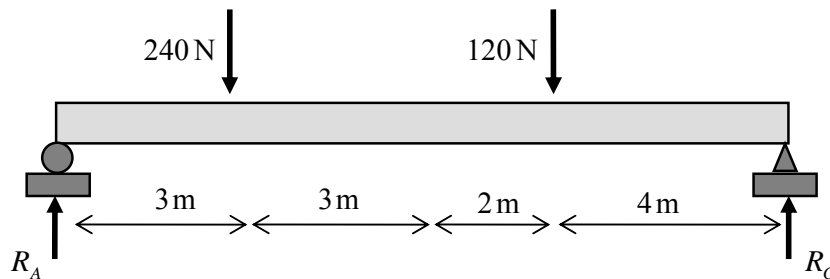$$R_A = 220 \text{ N}, \quad R_C = 140 \text{ N} \qquad (7.4.6)$$

240 N   120 N

R_A 3 m   3 m   2 m   4 m R_C

Figure 7.4.14: equivalent forces acting on the beam of Fig. 7.4.13

Referring again to Fig. 7.4.13, there are two distinct regions in the beam, that under the uniform load and that under the triangular distribution of load. The first case is considered in Fig. 7.4.15.
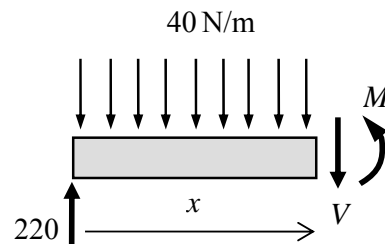

40 N/m

M

220   x   V

Figure 7.4.15: free body diagram of a section of a beam

The equilibrium equations give

$$V = 220 - 40x, \qquad M = 220x - 20x^2 \qquad \left(0 < x < 6\right) \qquad (7.4.7)$$

The region beneath the triangular distribution is shown in Fig. 7.4.16. Two possible approaches are illustrated: in Fig. 7.4.16a, the free body diagram consists of the complete length of beam to the left of the cross-section under consideration; in Fig. 7.4.16b, only the portion to the right is considered, with distance measured from the right hand end, as $12 - x$. The problem is easier to solve using the second option; from Fig. 7.4.16b then, with the equilibrium equations, one finds that

$$V = -140 + 10(12 - x)^2 / 3, \quad M = 140(12 - x) - 10(12 - x)^3 / 9 \quad (6 < x < 12) \quad (7.4.8)$$
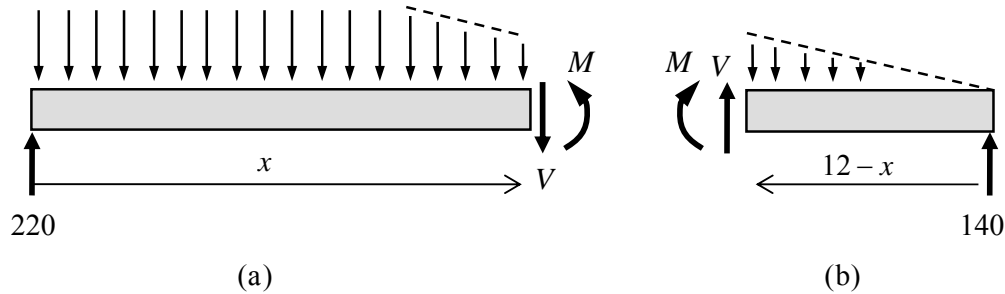

Figure 7.4.16: free body diagrams of sections of a beam

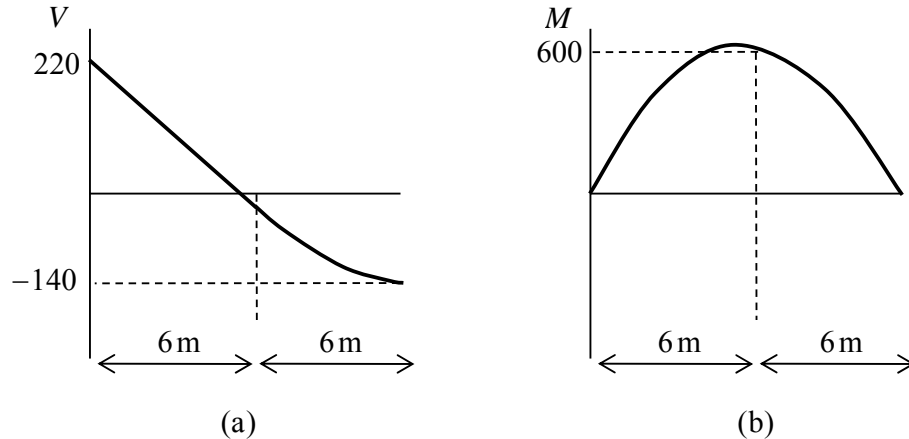The results are summarized in the shear force and bending moment diagrams of Fig. 7.4.17.

Figure 7.4.17: results of analysis; (a) shear force diagram, (b) bending moment diagram

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7.4.3   The Relationship between Loads, Shear Forces and Bending Moments

Relationships between the applied loads and the internal shear force and bending moment in a beam can be established by considering a small beam element, of width Δx, and

subjected to a distributed load $p(x)$ which varies along the section of beam, and which is *positive upward*, Fig. 7.4.18.
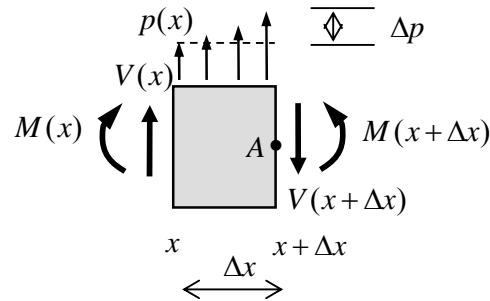

Figure 7.4.18: forces and moments acting on a small element of beam

At the left-hand end of the free body, at position x, the shear force, moment and distributed load have values $V(x)$, $M(x)$ and $p(x)$ respectively. On the right-hand end, at position $x + \Delta x$, their values are slightly different: $V(x + \Delta x)$, $M(x + \Delta x)$ and $p(x + \Delta x)$. Since the element is very small, the distributed load, even if it is varying, can be approximated by a *linear* variation over the element. The distributed load can therefore be considered to be a uniform distribution of intensity $p(x)$ over the length Δx together with a triangular distribution, 0 at x and Δp say, a *small* value, at $x + \Delta x$. Equilibrium of vertical forces then gives

$$V(x) + p(x)\Delta x + \frac{1}{2}\Delta p \Delta x - V(x + \Delta x) = 0$$
$$\rightarrow \frac{V(x + \Delta x) - V(x)}{\Delta x} = p(x) + \frac{1}{2}\Delta p \tag{7.4.9}$$

Now let the size of the element decrease towards zero. The left-hand side of Eqn. 7.4.9 is then the definition of the derivative, and the second term on the right-hand side tends to zero, so

$$\boxed{\frac{dV}{dx} = p(x)} \tag{7.4.10}$$

This relation can be seen to hold in Eqn. 7.4.7 and Fig. 7.4.17a, where the shear force over $0 < x < 6$ has a slope of -40 and the pressure distribution is uniform, of intensity $-40\,\text{N/m}$. Similarly, over $6 < x < 12$, the pressure decreases linearly and so does the slope in the shear force diagram, reaching zero slope at the end of the beam.

It also follows from 7.4.10 that the change in shear along a beam is equal to the area under the distributed load curve:

$$V(x_2) - V(x_1) = \int_{x_1}^{x_2} p(x)dx \tag{7.4.11}$$

Consider now moment equilibrium, by taking moments about the point *A* in Fig. 7.4.18:

$$-M(x) - V(x)\Delta x + M(x + \Delta x) - p(x)\Delta x \frac{\Delta x}{2} - \frac{1}{2}\Delta p \Delta x \frac{\Delta x}{3} = 0$$

$$\rightarrow \frac{M(x + \Delta x) - M(x)}{\Delta x} = V(x) + p(x)\frac{\Delta x}{2} + \Delta p \frac{\Delta x}{6}$$

(7.4.12)

Again, as the size of the element decreases towards zero, the left-hand side becomes a derivative and the second and third terms on the right-hand side tend to zero, so that

$$\boxed{\frac{dM}{dx} = V(x)}$$

(7.4.13)

This relation can be seen to hold in Eqns. 7.4.2-3, 7.4.5 and 7.4.7-8. It also follows from Eqn. 7.4.13 that the change in moment along a beam is equal to the area under the shear force curve:

$$M(x_2) - M(x_1) = \int_{x_1}^{x_2} V(x)dx$$

(7.4.14)

7.4.4 Deformation and Flexural Stresses in Beams

The moment at any given cross-section of a beam is due to a distribution of normal stress, or **flexural stress** (or **bending stress**) across the section (see Fig. 7.4.4). As mentioned, the stresses to one side of the neutral axis are tensile whereas on the other side of the neutral axis they are compressive. To determine the distribution of normal stress over the section, one must determine the precise location of the neutral axis, and to do this one must consider the *deformation* of the beam.

Apart from the assumption of there being a longitudinal plane of symmetry and a neutral axis along which material fibres do not extend, the following two assumptions will be made concerning the deformation of a beam:

1. Cross-sections which are plane and are perpendicular to the axis of the undeformed beam remain plane and remain perpendicular to the deflection curve of the deformed beam. In short: "plane sections remain plane". This is illustrated in Fig. 7.4.19. It will be seen later that this assumption is a valid one provided the beam is sufficiently long and slender.

2. Deformation in the vertical direction, i.e. the transverse strain ε_{yy}, may be neglected in deriving an expression for the longitudinal strain ε_{xx}. This assumption is summarised in the deformation shown in Fig. 7.4.20, which shows an element of length *l* and height *h* undergoing transverse and longitudinal strain.
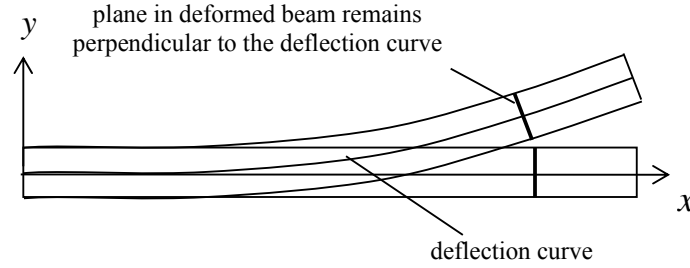
Figure 7.4.19: plane sections remain plane in the elementary beam theory

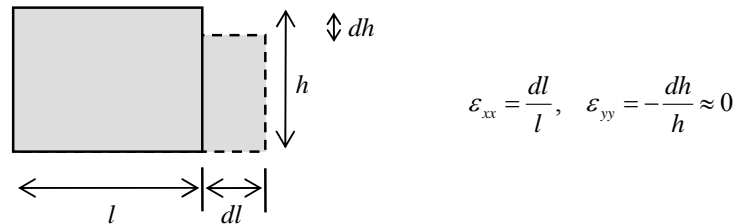$$\varepsilon_{xx} = \frac{dl}{l}, \quad \varepsilon_{yy} = -\frac{dh}{h} \approx 0$$

Figure 7.4.20: transverse strain is neglected in the elementary beam theory

With these assumptions, consider now the element of beam shown in Fig. 7.4.21. Here, two material fibres ab and pq, of length Δx in the undeformed beam, deform to $a'b'$ and $p'q'$. The deflection curve has a radius of curvature R. The above two assumptions imply that, referring to the figure:

$$\angle p'a'b' = \angle a'b'q' = \pi/2 \qquad \text{(assumption 1)}$$
$$|ap| = |a'p'|, \quad |bq| = |b'q'| \qquad \text{(assumption 2)} \tag{7.4.15}$$

Since the fibre ab is on the neutral axis, by definition $|a'b'| = |ab|$. However the fibre pq, a distance y from the neutral axis, extends in length from Δx to length $\Delta x'$. The longitudinal strain for this fibre is

$$\varepsilon_{xx} = \frac{\Delta x' - \Delta x}{\Delta x} = \frac{(R-y)\Delta\theta - R\Delta\theta}{R\Delta\theta} = -\frac{y}{R} \tag{7.4.16}$$

As one would expect, this relation implies that a small R (large curvature) is related to a large strain and a large R (small curvature) is related to a small strain. Further, for $y > 0$ (above the neutral axis), the strain is negative, whereas if $y < 0$ (below the neutral axis), the strain is positive[4], and the variation across the cross-section is linear.

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[4] this is under the assumption that R is positive, which means that the beam is concave up; a negative R implies that the centre of curvature is below the beam

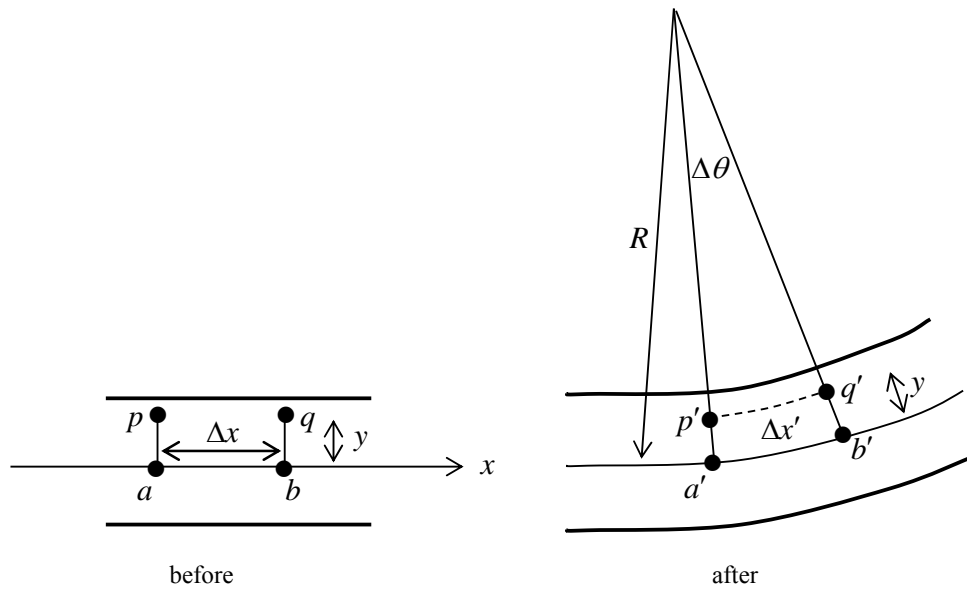before   after

Figure 7.4.21: deformation of material fibres in an element of beam

To relate this deformation to the stresses arising in the beam, it is necessary to postulate the stress-strain law for the material out of which the beam is made. Here, it is assumed that the beam is isotropic linear elastic[5].

The beam is a three-dimensional object, and so will in general experience a fairly complex three-dimensional stress state. We will show in what follows that a simple one-dimensional approximation, $\sigma_{xx} = E\varepsilon_{xx}$, whilst disregarding all other stresses and strains, will be sufficiently accurate for our purposes.

Since there are no forces acting in the z direction, the beam is in a state of plane stress, and the stress-strain equations are (see Eqns. 6.1.10)

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$$

$$\varepsilon_{zz} = -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right] \tag{7.4.17}$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}, \qquad \varepsilon_{xz} = \varepsilon_{yz} = 0$$

Yet another assumption is now made, that the transverse normal stresses, σ_{yy}, may be neglected in comparison with the flexural stresses σ_{xx}. This is similar to the above assumption #2 concerning the deformation, where the transverse normal strain was neglected in comparison with the longitudinal strain. It might seem strange at first that the transverse stress is neglected, since all loads are in the transverse direction. However,

---

[5] the beam theory can be extended to incorporate more complex material models (constitutive equations)

just as the tangential stresses are much larger than the radial stresses in the pressure vessel, it is found that the longitudinal stresses in a beam are very much greater than the transverse stresses. With this assumption, the first of Eqn. 7.4.17 reduces to a one-dimensional equation:

$$\varepsilon_{xx} = \sigma_{xx} / E \tag{7.4.18}$$

and, from Eqn. 7.4.16, dropping the subscripts on σ,

$$\sigma = -\frac{E}{R} y \tag{7.4.19}$$

Finally, the resultant force of the normal stress distribution over the cross-section must be zero, and the resultant moment of the distribution is M, leading to the conditions

$$0 = \int_A \sigma dA = -\frac{E}{R} \int_A y dA$$

$$M = -\int_A \sigma y dA = \frac{E}{R} \int_A y^2 dA = -\frac{\sigma}{y} \int_A y^2 dA \tag{7.4.20}$$

and the integration is over the complete cross-sectional area A. The minus sign in the second of these equations arises because a positive moment and a positive y imply a compressive (negative) stress (see Fig. 7.4.4).

The quantity $\int_A y dA$ is the first moment of area about the neutral axis, and is equal to $\bar{y} A$, where $\bar{y}$ is the centroid of the section (see, for example, §3.2.1). Note that the horizontal component ("in-out of the page") of the centroid will always be at the centre of the beam due to the symmetry of the beam about the plane of bending. Since the first moment of area is zero, it follows that $\bar{y} = 0$: *the neutral axis passes through the centroid of the cross-section*.

The quantity $\int_A y^2 dA$ is called the **second moment of area** or the **moment of inertia** about the neutral axis, and is denoted by the symbol I. It follows that the flexural stress is related to the moment through

$$\boxed{\sigma = -\frac{My}{I}} \qquad \textbf{Flexural stress in a beam} \tag{7.4.21}$$

This is one of the most famous and useful formulas in mechanics.

The Moment of Inertia

The moment of inertia depends on the shape of a beam's cross-section. Consider the important case of a rectangular cross section. Before determining the moment of inertia one must locate the centroid (neutral axis). Due to symmetry, the neutral axis runs

through the centre of the cross-section. To evaluate I for a rectangle of height h and width b, consider a small strip of height dy at location y, Fig. 7.4.22. Then

$$I = \int_A y^2 dA = b \int_{-h/2}^{+h/2} y^2 dy = \frac{bh^3}{12} \tag{7.4.22}$$

This relation shows that the "taller" the cross-section, the larger the moment of inertia, something which holds generally for I. Further, the larger is I, the smaller is the flexural stress, which is always desirable.
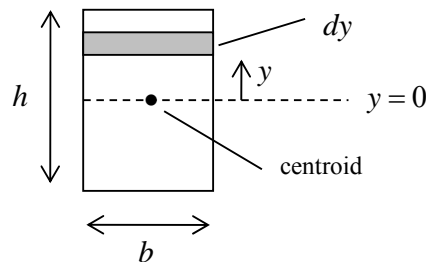

Figure 7.4.22: Evaluation of the moment of inertia for a rectangular cross-section

For a circular cross-section with radius R, consider Fig. 7.4.23. The moment of inertia is then

$$I = \int_A y^2 dA = \int_0^{2\pi} \int_0^R r^3 \sin^2 \theta \, dr \, d\theta = \frac{\pi R^4}{4} \tag{7.4.23}$$
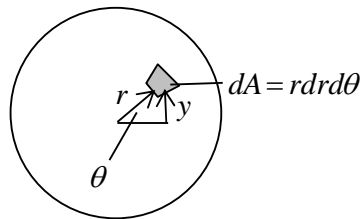

Figure 7.2.23: Moment of inertia for a circular cross-section

Example

Consider the beam shown in Fig. 7.4.24. It is loaded symmetrically by two concentrated forces each of magnitude 100N and has a circular cross-section of radius 100mm. The reactions at the two supports are found to be 100N. Sectioning the beam to the left of the forces, and then to the right of the first force, one finds that

$$\begin{aligned} V &= 100, & M &= 100x & (0 < x < 250) \\ V &= 0, & M &= 25000 & (250 < x < l/2) \end{aligned} \tag{7.4.24}$$
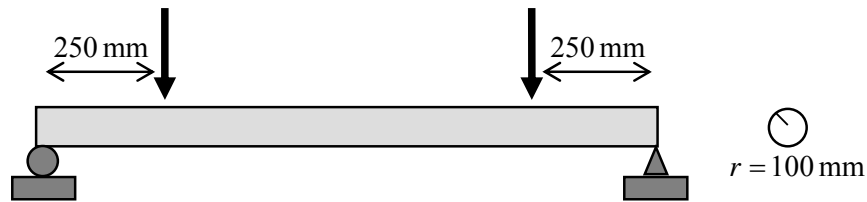
where *l* is the length of the beam.

Figure 7.4.24: a loaded beam with circular cross-section

The maximum tensile stress is then

$$\sigma_{max} = -\frac{M_{max}(-y_{max})}{I} = \frac{25000r}{\pi r^4/4} = 31.8\,\text{MPa} \qquad (7.4.25)$$

and occurs at all sections between the two loads (at the base of the beam).

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7.4.5 Shear Stresses in Beams

In the derivation of the flexural stress formula, Eqn. 7.4.21, it was assumed that plane sections remain plane. This implies that there is no shear strain and, for an isotropic elastic material, no shear stress, as indicated in Fig. 7.4.25.
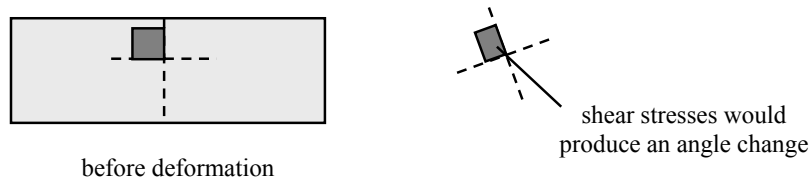

before deformation

shear stresses would produce an angle change

Figure 7.4.25: a section of beam before and after deformation

This fact will now be ignored, and an expression for the shear stress τ within a beam will be developed. It is implicitly assumed that this shear stress has little effect on the (calculation of the) flexural stress.

As in Fig. 7.4.18, consider the equilibrium of a thin section of beam, as shown in Fig. 7.4.26. The beam has *rectangular* cross-section (although the theory developed here is strictly for rectangular cross sections only, it can be used to give approximate shear stress values in any beam with a plane of symmetry). Consider the equilibrium of a section *of* this section, at the upper surface of the beam, shown hatched in Fig. 7.4.26. The stresses acting on this section are as shown. Again, the normal stress is compressive at the surface, consistent with the sign convention for a positive moment. Note that there are no shear stresses acting at the surface – there may be distributed normal loads or forces acting at the surface but, for clarity, these are not shown, and they are not necessary for the following calculation.

From equilibrium of forces in the horizontal direction of the surface section:

$$\left[-\int_A \sigma dA \right]_x + \left[\int_A \sigma dA \right]_{x+\Delta x} + \tau b \Delta x = 0 \qquad (7.4.26)$$

The third term on the left here assumes that the shear stress is uniform over the section – this is similar to the calculations of §7.4.3 – for a very small section, the variation in stress is a small term and may be neglected. Using the bending stress formula, Eqn. 7.4.21,

$$-\int_A \frac{M(x+\Delta x) - M(x)}{\Delta x} \frac{y}{I} dA + \tau b = 0 \qquad (7.4.27)$$

and, with Eqn. 7.4.13, as $\Delta x \to 0$,

$$\boxed{\tau = \frac{VQ}{Ib}} \qquad \textbf{Shear stress in a beam} \qquad (7.4.28)$$

where Q is the first moment of area $\int_A y dA$ of the *surface section* of the cross-section.
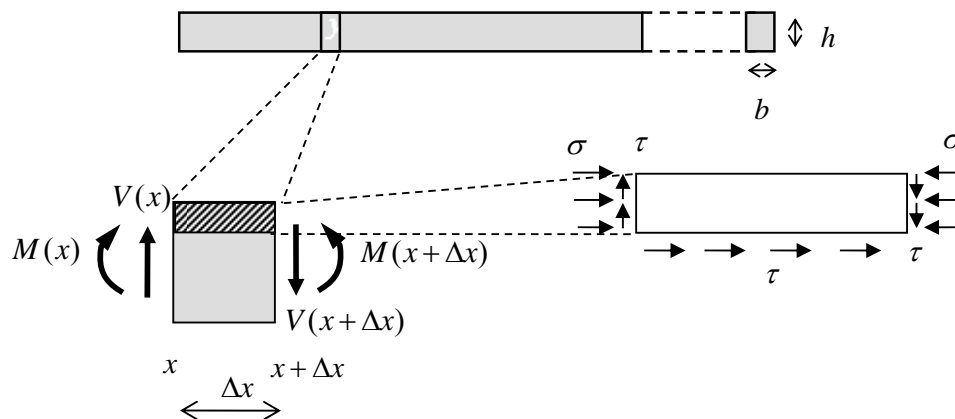

Figure 7.4.26: stresses and forces acting on a small section of material at the surface of a beam

As mentioned, this formula 7.4.28 can be used as an approximation of the shear stress in a beam of arbitrary cross-section, in which case b can be regarded as the depth of the beam at that section. For the rectangular beam, one has

$$Q = b\int_y^{h/2} y dy = \frac{b}{2}\left(\frac{h^2}{4} - y^2 \right) \qquad (7.4.29)$$

so that

$$\tau = \frac{6V}{bh^3}\left(\frac{h^2}{4} - y^2\right)$$ (7.4.30)

The maximum shear stress in the cross-section arises at the neutral surface:

$$\tau_{max} = \frac{3V}{2bh} = \frac{3V}{2A}$$ (7.4.31)

and the shear stress dies away towards the upper and lower surfaces. Note that the average shear stress over the cross-section is V/A and the maximum shear stress is 150% of this value.

Finally, since the shear stress on a vertical cross-section has been evaluated, the shear stress on a longitudinal section has been evaluated, since the shear stresses on all four sides of an element are the same, as in Fig.7.4.6.

Example

Consider the simply supported beam loaded by a concentrated force shown in Fig. 7.4.27. The cross-section is rectangular with height $100\,mm$ and width $50\,mm$. The reactions at the supports are $5\,kN$ and $15\,kN$. To the left of the load, one has $V = 5\,kN$ and $M = 5x\,kNm$. To the right of the load, one has $V = -15\,kN$ and $M = 30 - 15x\,kNm$.

The maximum shear stress will occur along the neutral axis and will clearly occur where V is largest, so anywhere to the right of the load:

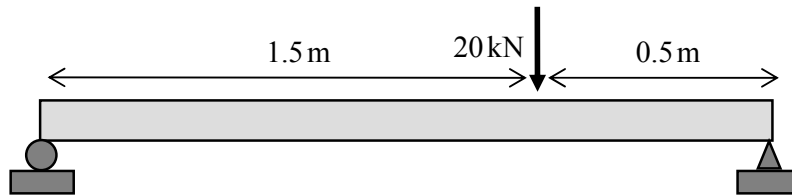$$\tau_{max} = \frac{3V_{max}}{2A} = 4.5\,MPa$$ (7.4.32)

Figure 7.4.27: a simply supported beam

As an example of general shear stress evaluation, the shear stress at a point 25 mm below the top surface and 1 m in from the left-hand end is, from Eqn 7.4.30, $\tau = +1.125\,MPa$. The shear stresses acting on an element at this location are shown in Fig. 7.4.28.
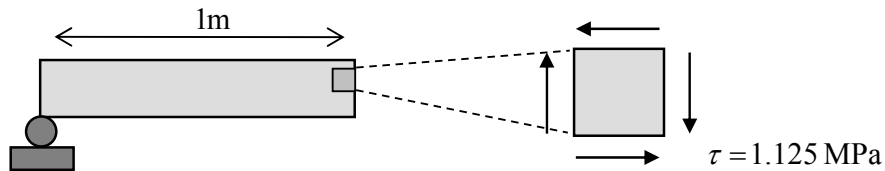
Figure 7.4.28: shear stresses acting at a point in the beam

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7.4.6 Approximate nature of the beam theory

The beam theory is only an approximate theory, with a number of simplifications made to the full equations of elasticity. A more advanced (and exact) mechanics treatment of the beam problem would not make any assumptions regarding plane sections remaining plane, etc. The accuracy of the beam theory can be explored by comparing the beam theory results with the results of the more exact theory.

When a beam is in **pure bending**, that is when the shear force is everywhere zero, the full elasticity solution shows that plane sections *do* actually remain plane and the beam theory is exact. For more complex loadings, plane sections *do* actually deform. For example, it can be shown that the initially plane sections of a cantilever subjected to an end force, Fig. 7.4.29, do not remain plane. Nevertheless, the beam theory prediction for normal and shear stress is exact in this simple case.
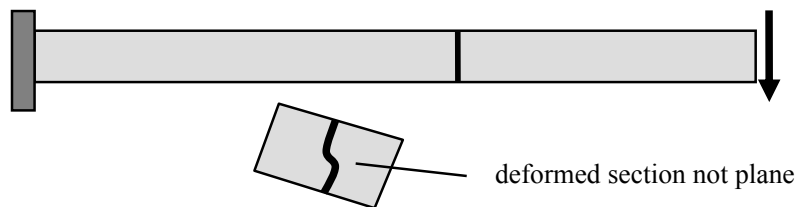

deformed section not plane

Figure 7.4.29: a cantilevered beam loaded by a force and moment

Consider next a cantilevered beam of length l and rectangular cross section, height h and width b, subjected to a uniformly distributed load p. With x measured from the cantilevered end, the shear force and moment are given by $V = p(l-x)$ and $M = (pl^2/2)\left(-1+2x/l-(x/l)^2\right)$. The shear stress is

$$\tau = \frac{6p}{bh^3}\left(\frac{h^2}{4}-y^2\right)(l-x) \tag{7.4.33}$$

and the flexural stresses at the cantilevered end, at the upper surface, are

$$\frac{\sigma}{p} = \frac{3}{b}\left(\frac{l}{h}\right)^2 \tag{7.4.34}$$

The solution for shear stress, Eqn 7.4.33, turns out to be exact; however, the exact solution corresponding to Eqn 7.4.34 is[6]

$$\frac{\sigma}{p} = \frac{1}{b}\left[3\left(\frac{l}{h}\right)^2 - \frac{1}{5}\right]$$
(7.4.35)

It can be seen that the beam theory is a good approximation for the case when l/h is large, in which case the term 1/5 is negligible.

Following this type of analysis, a general rule of thumb is this: for most configurations, the elementary beam theory formulae for flexural stress and transverse shear stress are accurate to within about 3% for beams whose length-to-height ratio is greater than about 4.

7.4.7   Beam Deflection

Consider the deflection curve of a beam. The displacement of the neutral axis is denoted by v, positive upwards, as in Fig. 7.4.30. The slope at any point is then given by the first derivative, dv/dx.

For any type of material, provided the slope of the deflection curve is small, it can be shown that the radius of curvature R is related to the second derivative d^2v/dx^2 through (see the Appendix to this section, §7.4.10)

$$\frac{1}{R} = \frac{d^2v}{dx^2}$$
(7.4.36)

and for this reason d^2v/dx^2 is called the **curvature** of the beam. Using Eqn. 7.4.19, $\sigma = -Ey/R$, and the flexural stress expression, Eqn. 7.4.21, $\sigma = -My/I$, one has the **moment-curvature equation**

$$\boxed{M(x) = EI\frac{d^2v}{dx^2}}\qquad \textbf{moment-curvature equation}\qquad (7.4.37)$$
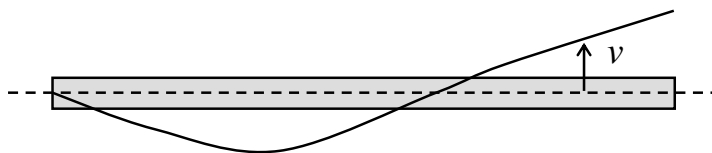

Figure 7.4.30: the deflection of a beam

With the moment known, this differential equation can be integrated twice to obtain the deflection. Boundary conditions must be supplied to obtain constants of integration.

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[6] this can be derived using the Stress Function method discussed in Book 2, section 3.2

Example

Consider the cantilevered beam of length L shown in Fig. 7.4.31, subjected to an end-force F and end-moment M_0. The moment is found to be $M(x) = F(L-x) + M_0$, with x measured from the clamped end. The moment-curvature equation is then

$$EI\frac{d^2v}{dx^2} = (FL + M_0) - Fx$$

$$\rightarrow \quad EI\frac{dv}{dx} = (FL + M_0)x - \frac{1}{2}Fx^2 + C_1 \qquad (7.4.38)$$

$$\rightarrow \quad EIv = \frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3 + C_1x + C_2$$

The boundary conditions are that the displacement and slope are both zero at the clamped end, from which the two constant of integration can be obtained:

$$
\begin{aligned}
v(0) = 0 &\quad \rightarrow \quad C_2 = 0 \\
v'(0) = 0 &\quad \rightarrow \quad C_1 = 0
\end{aligned}
\qquad (7.4.39)
$$
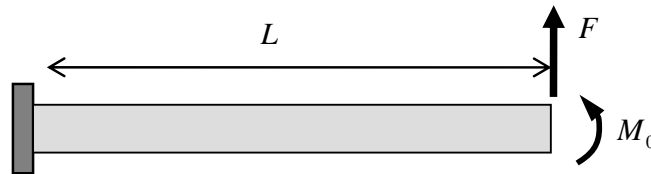
Figure 7.4.31: a cantilevered beam loaded by an end-force and moment

The slope and deflection are therefore

$$v = \frac{1}{EI}\left[\frac{1}{2}(FL + M_0)x^2 - \frac{1}{6}Fx^3\right], \qquad \frac{dv}{dx} = \frac{1}{EI}\left[(FL + M_0)x - \frac{1}{2}Fx^2\right] \qquad (7.4.40)$$

The maximum deflection occurs at the end, where

$$v(L) = \frac{1}{EI}\left[\frac{1}{2}M_0L^2 + \frac{1}{3}FL^3\right] \qquad (7.4.41)$$

∎

The term EI in Eqns. 7.4.40-41 is called the **flexural rigidity**, since it is a measure of the resistance of the beam to deflection.

Example

Consider the simply supported beam of length L shown in Fig. 7.4.32, subjected to a uniformly distributed load p over half its length. In this case, the moment is given by

$$M(x) = \begin{cases} \dfrac{3}{8}pLx - \dfrac{1}{2}px^2 & 0 < x < \dfrac{L}{2} \\[4mm] \dfrac{1}{8}pL(L-x) & \dfrac{L}{2} < x < L \end{cases} \tag{7.4.42}$$
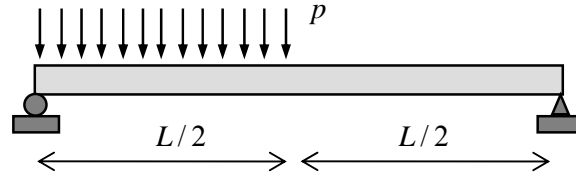

Figure 7.4.32: a simply supported beam subjected to a uniformly distributed load over half its length

It is necessary to apply the moment-curvature equation to each of the two regions $0 < x < L/2$ and $L/2 < x < L$ separately, since the expressions for the moment in these regions differ. Thus there will be four constants of integration:

$$EI\frac{d^2v}{dx^2} = \frac{3}{8}pLx - \frac{1}{2}px^2 \qquad\qquad EI\frac{d^2v}{dx^2} = \frac{1}{8}pL^2 - \frac{1}{8}pLx$$

$$\rightarrow EI\frac{dv}{dx} = \frac{3}{16}pLx^2 - \frac{1}{6}px^3 + C_1 \qquad \rightarrow EI\frac{dv}{dx} = \frac{1}{8}pL^2x - \frac{1}{16}pLx^2 + D_1$$

$$\rightarrow EIv = \frac{3}{48}pLx^3 - \frac{1}{24}px^4 + C_1x + C_2 \qquad \rightarrow EIv = \frac{1}{16}pL^2x^2 - \frac{1}{48}pLx^3 + D_1x + D_2$$

$$\tag{7.4.43}$$

The boundary conditions are: (i) no deflection at roller support, $v(0) = 0$, from which one finds that $C_2 = 0$, and (ii) no deflection at pin support, $v(L) = 0$, from which one finds that $D_2 = -pL^4/24 - D_1L$. The other two necessary conditions are the **continuity conditions** where the two solutions meet. These are that (i) the deflection of both solutions agree at $x = L/2$ and (ii) the slope of both solutions agree at $x = L/2$. Using these conditions, one finds that

$$C_1 = -\frac{9pL^3}{384}, \quad C_2 = -\frac{17pL^3}{384} \tag{7.4.44}$$

so that

$$v = \begin{cases} \dfrac{wL^4}{384EI}\left[-9\left(\dfrac{x}{L}\right) + 24\left(\dfrac{x}{L}\right)^3 - 16\left(\dfrac{x}{L}\right)^4\right] & 0 < x < \dfrac{L}{2} \\[5mm] \dfrac{wL^4}{384EI}\left[1 - 17\left(\dfrac{x}{L}\right) + 24\left(\dfrac{x}{L}\right)^2 - 8\left(\dfrac{x}{L}\right)^3\right] & \dfrac{L}{2} < x < L \end{cases} \tag{7.4.45}$$

The deflection is shown in Fig. 7.4.33. Note that the maximum deflection occurs in $0 < x < L/2$; it can be located by setting $dv/dx = 0$ there and solving.
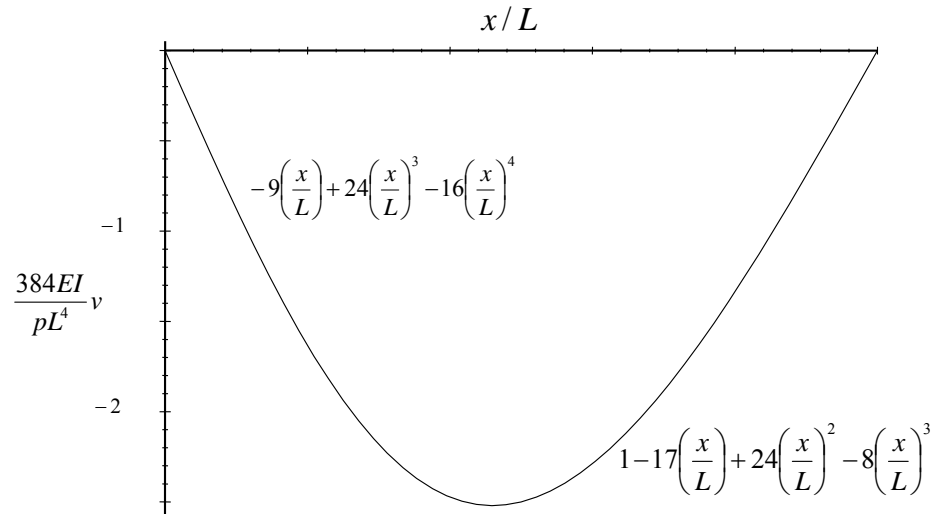
$$x/L$$

$$-9\left(\frac{x}{L}\right) + 24\left(\frac{x}{L}\right)^3 - 16\left(\frac{x}{L}\right)^4$$

-1

$$\frac{384EI}{pL^4}v$$

-2

$$1 - 17\left(\frac{x}{L}\right) + 24\left(\frac{x}{L}\right)^2 - 8\left(\frac{x}{L}\right)^3$$

Figure 7.4.33: deflection of a beam

∎

7.4.8 Statically Indeterminate Beams

Consider the beam shown in Fig. 7.4.34. It is cantilevered at one end and supported by a roller at its other end. A moment is applied at its centre. There are three unknown reactions in this problem, the reaction force at the roller and the reaction force and moment at the built-in end. There are only two equilibrium equations with which to determine these three unknowns and so it is not possible to solve the problem from equilibrium considerations alone. The beam is therefore statically indeterminate (see the end of section 2.3.3).

M_0
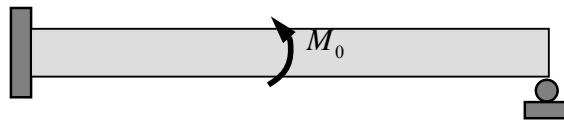
Figure 7.4.34: a cantilevered beam supported also by a roller

More examples of statically indeterminate beam problems are shown in Fig. 7.4.35. To solve such problems, one must consider the deformation of the beam. The following example illustrates how this can be achieved.
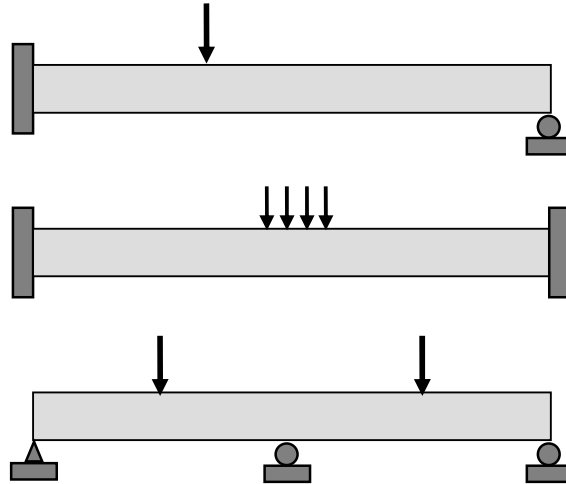
Figure 7.4.35: examples of statically indeterminate beams

Example

Consider the beam of length L shown in Fig. 7.4.36, cantilevered at end A and supported by a roller at end B. A moment M_0 is applied at B.
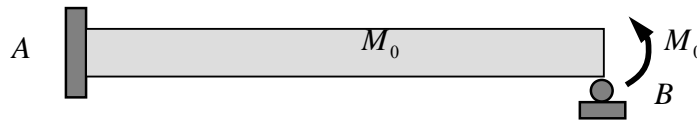

Figure 7.4.36: a statically indeterminate beam

The moment along the beam can be expressed in terms of the *unknown* reaction force at end B: $M(x) = R_B(L - x) + M_0$. As before, one can integrate the moment-curvature equation:

$$EI \frac{d^2v}{dx^2} = R_B(L - x) + M_0$$

$$\rightarrow EI \frac{dv}{dx} = (R_B L + M_0)x - \frac{1}{2}R_B x^2 + C_1 \qquad (7.4.46)$$

$$\rightarrow EIv = \frac{1}{2}(R_B L + M_0)x^2 - \frac{1}{6}R_B x^3 + C_1 x + C_2$$

There are three boundary conditions, two to determine the constants of integration and one can be used to determine the unknown reaction R_B. The boundary conditions are (i) $v(0) = 0 \rightarrow C_2 = 0$, (ii) $dv/dx(0) = 0 \rightarrow C_1 = 0$ and (iii) $v(L) = 0$ from which one finds that $R_B = -3M_0/2L$. The slope and deflection are therefore

$$v = \frac{M_B L^2}{4EI}\left[\left(\frac{x}{L}\right)^3 - \left(\frac{x}{L}\right)^2\right]$$

$$\frac{dv}{dx} = \frac{M_B L}{4EI}\left[3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)\right]$$

(7.4.47)

One can now return to the equilibrium equations to find the remaining reactions acting on the beam, which are $R_A = -R_B$ and $M_A = M_0 + LR_B$

∎

7.4.9 The Three-point Bending Test

The 3-point bending test is a very useful experimental procedure. It is used to gather data on materials which are subjected to bending in service. It can also be used to get the Young's Modulus of a material for which it might be more difficult to get *via* a tension or other test.

A mouse bone is shown in the standard 3-point bend test apparatus in Fig. 7.4.37a. The idealised beam theory model of this test is shown in Fig. 7.4.37b. The central load is F, so the reactions at the supports are $F/2$. The moment is zero at the supports, varying linearly to a maximum $FL/4$ at the centre.

(a) (b)
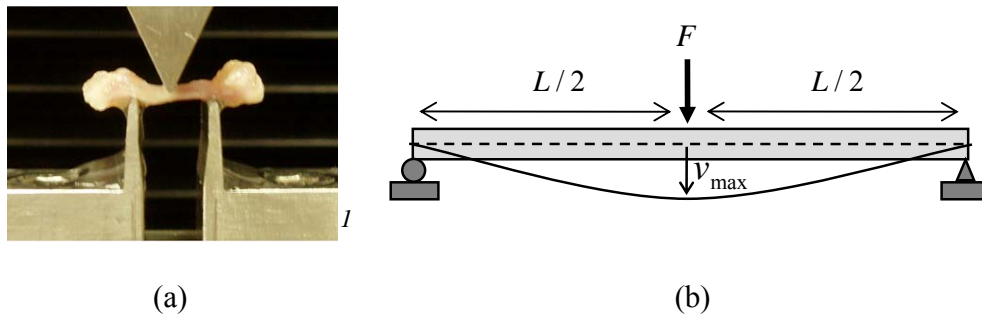
**Figure 7.4.37: the three-point bend test; (a) a mouse bone specimen, (b) idealised model**

The maximum flexural stress then occurs at the outer fibres at the centre of the beam: for a circular cross-section, $\sigma_{max} = FL/\pi R^3$. Integrating the moment-curvature equation, and using the fact that the deflection is zero at the supports and, from symmetry, the slope is zero at the centre, the maximum deflection is seen to be $v_{max} = FL^3/12\pi R^4 E$. If one plots the load F against the deflection v_{max}, one will see a straight line (initially, before the elastic limit is reached); let the slope of this line be $\hat{E}$. The Young's modulus can then be evaluated through

$$E = \frac{L^3}{12\pi R^4}\hat{E}$$

(7.4.48)

With $\sigma = E\varepsilon$, the maximum strain is $\varepsilon_{max} = FL / \pi ER^3 = 12Rv_{max} / L^2$. By carrying the test on beyond the elastic limit, the strength of the material at failure can be determined.

7.4.10 Problems

1. The simply supported beam shown below carries a vertical load that increases uniformly from zero at the left end to a maximum value of 9 kN/m at the right end. Draw the shearing force and bending moment diagrams

9 kN/m

6 m
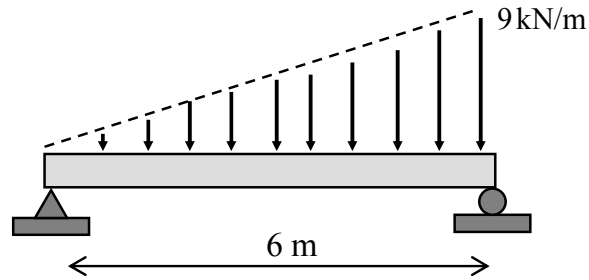
2. The beam shown below is simply supported at two points and overhangs the supports at each end. It is subjected to a uniformly distributed load of 4 kN/m as well as a couple of magnitude 8 kN m applied to the centre. Draw the shearing force and bending moment diagrams

8 kN m

4 kN/m

1 m   2 m   2 m   1 m
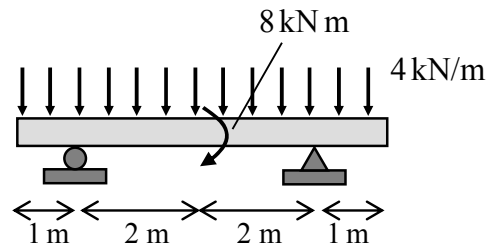
3. Evaluate the centroid of the beam cross-section shown below (all measurements in mm)

75   75

50

225

50

50   100
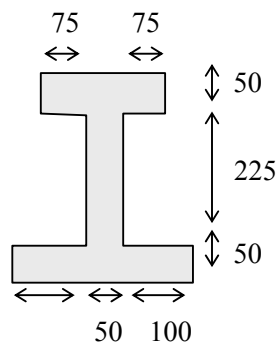
4. Determine the maximum tensile and compressive stresses in the following beam (it has a rectangular cross-section with height 75 mm and depth 50 mm)

1kN

5 m
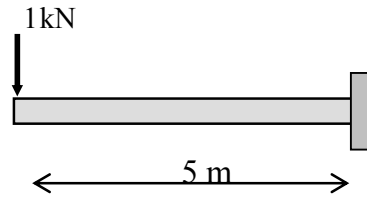
5. Consider the cantilever beam shown below. Determine the maximum shearing stress in the beam and determine the shearing stress 25 mm from the top surface of the beam at a section adjacent to the supporting wall. The cross-section is the "T" shape shown, for which $I = 40 \times 10^6 \text{ mm}^4$.

[note: use the shear stress formula derived for rectangular cross-sections – as mentioned above, in this formula, b is the thickness of the beam *at the point where the shear stress is being evaluated*]

50 kN

2 m

50

125

50

200
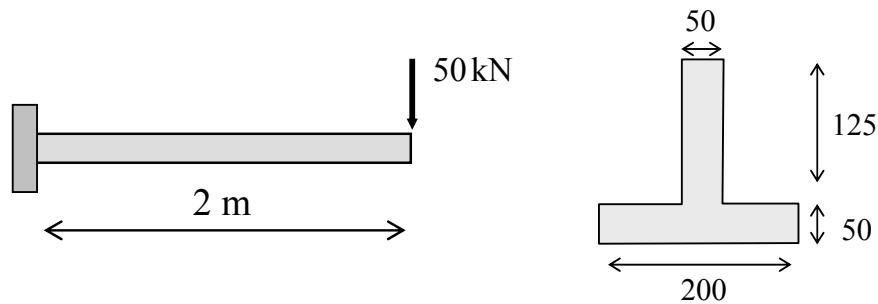
6. Obtain an expression for the maximum deflection of the simply supported beam shown here, subject to a uniformly distributed load of $w \, \text{N/m}$.

$w \, \text{N/m}$

L
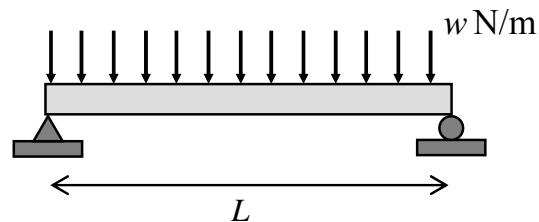
7. Determine the equation of the deflection curve for the cantilever beam loaded by a concentrated force P as shown below.

a

P

L
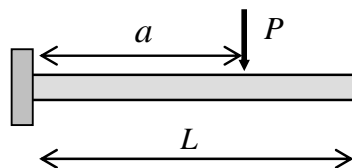
8. Determine the reactions for the following uniformly loaded beam clamped at both ends.
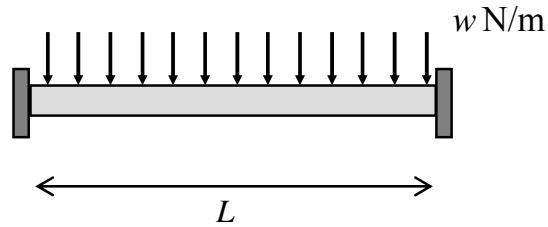
7.4.11 Appendix to §7.4

Curvature of the deflection curve

Consider a deflection curve with deflection $v(x)$ and radius of curvature $R(x)$, as shown in the figure below. Here, *deflection* is the transverse displacement (in the y direction) of the points that lie along the axis of the beam. A relationship between $v(x)$ and $R(x)$ is derived in what follows.
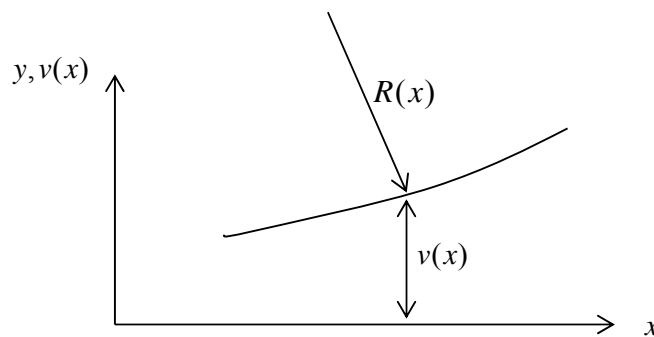
First, consider a curve (arc) s. The tangent to some point p makes an angle ψ with the x – axis, as shown below. As one move along the arc, ψ changes.
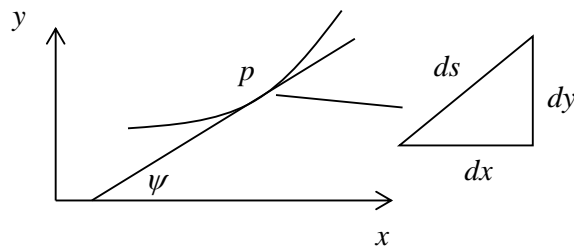
Define the **curvature** κ of the curve to be the rate at which ψ increases relative to s,

$$\kappa = \frac{d\psi}{ds}$$

Thus if the curve is very "curved", ψ is changing rapidly as one moves along the curve (as one increase s) and the curvature will be large.

From the above figure,

$$\tan\psi = \frac{dy}{dx}, \quad \frac{ds}{dx} = \frac{\sqrt{(dx)^2 + (dy)^2}}{dx} = \sqrt{1 + (dy/dx)^2} \; ,$$

so that

$$\kappa = \frac{d\psi}{ds} = \frac{d\psi}{dx}\frac{dx}{ds} = \frac{d(\arctan(dy/dx))}{dx}\frac{dx}{ds} = \frac{1}{1+(dy/dx)^2}\frac{d^2y}{dx^2}\frac{dx}{ds}$$

$$= \frac{d^2y/dx^2}{\left[1+(dy/dx)^2\right]^{3/2}}$$

Finally, it will be shown that the curvature is simply the reciprocal of the radius of curvature. Draw a circle to the point p with radius R. Arbitrarily measure the arc length s from the point c, which is a point on the circle such that $\angle cop = \psi$. Then arc length $s = R\psi$, so that

$$\kappa = \frac{d\psi}{ds} = \frac{1}{R}$$
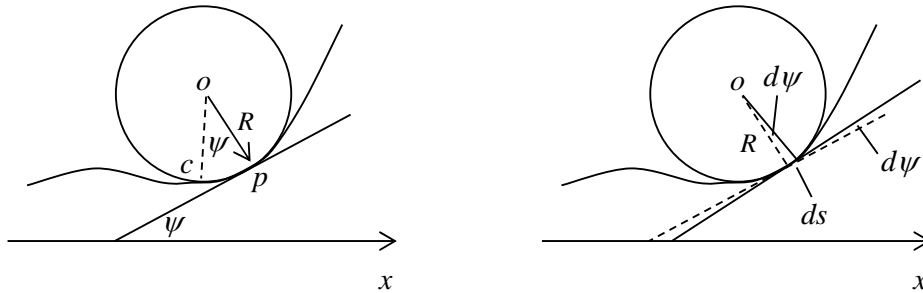
Thus

$$\frac{1}{R} = \frac{\dfrac{d^2v}{dx^2}}{\left[1+\left(\dfrac{dv}{dx}\right)^2\right]^{\frac{3}{2}}}$$

If one assumes now that the slopes of the deflection curve are small, then $dv/dx \ll 1$ and

$$\frac{1}{R} \approx \frac{d^2v}{dx^2}$$

Images used:
1.