

7.1 One Dimensional Axial Deformations

In this section, a specific simple geometry is considered, that of a long and thin straight component loaded in such a way that it deforms in the axial direction only. The x -axis is taken as the longitudinal axis, with the cross-section lying in the $x - y$ plane, Fig. 7.1.1.



Figure 7.1.1: A slender straight component; (a) longitudinal axis, (b) cross-section

7.1.1 Basic relations for Axial Deformations

Any static analysis of a structural component involves the following three considerations:

- (1) constitutive response
- (2) kinematics
- (3) equilibrium

In this Chapter, it is taken for (1) that the material responds as an isotropic linear elastic solid. It is assumed that the only significant stresses and strains occur in the axial direction, and so the stress-strain relations 6.1.8-9 reduce to the one-dimensional equation $\sigma_{xx} = E\varepsilon_{xx}$ or, dropping the subscripts,

$$\sigma = E\varepsilon \quad (7.1.1)$$

Kinematics (2), the study of deformation, was the subject of Chapter 4. In the theory developed here, known as **axial deformation**, it is assumed that the axis of the component remains straight and that cross-sections that are initially perpendicular to the axis remain perpendicular after deformation. This implies that, although the strain might vary along the axis, it remains *constant over any cross section*. The axial strain occurring over any section is defined by Eqn. 4.1.2,

$$\varepsilon = \frac{L - L_0}{L_0} \quad (7.1.2)$$

This is illustrated in Fig. 7.1.2, which shows a (shaded) region undergoing a compressive (negative) strain.

Recall that individual particles/points undergo displacements whereas regions/line-elements undergo strain. In Fig. 7.1.2, the particle originally at A has undergone a displacement $u(A)$ whereas the particle originally at B has undergone a displacement $u(B)$. From Fig. 7.1.2, another way of expressing the strain in the shaded region is (see Eqn. 4.1.3)

$$\varepsilon = \frac{u(B) - u(A)}{L_0} \quad (7.1.3)$$

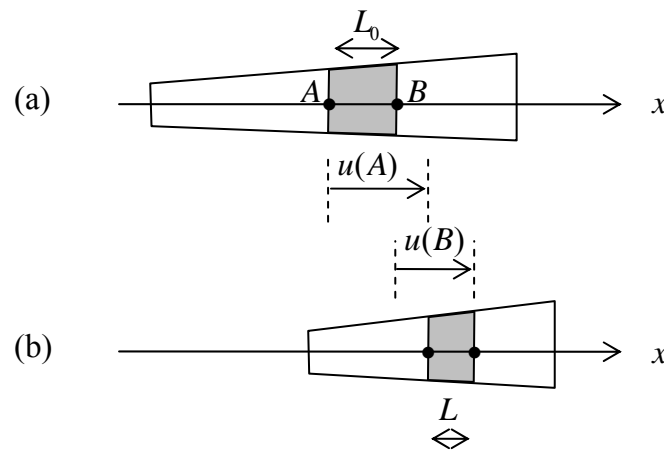


Figure 7.1.2: axial strain; (a) before deformation, (b) after deformation

Both displacements $u(A)$ and $u(B)$ of Fig. 7.1.2 are *positive*, since the particles displace in the positive x direction – if they moved to the left, for consistency, one would say they underwent *negative* displacements. Further, positive stresses are as shown in Fig. 7.1.3a and negative stresses are as shown in Fig. 7.1.3b. From Eqn. 7.1.1, a positive stress implies a positive strain (lengthening) and a compressive stress implies a negative strain (contracting)

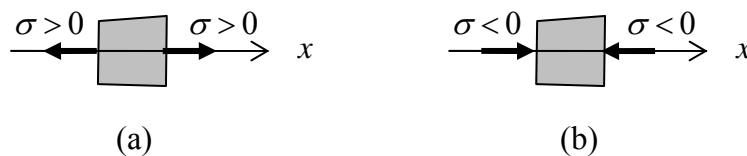


Figure 7.1.3: Stresses arising in the slender component; (a) positive (tensile) stress, (b) negative (compressive) stress

Equilibrium, (3), will be considered in the individual examples below.

Note that, in the previous Chapter, problems were solved using only the stress-strain law (1). Kinematics (2) and equilibrium (3) were not considered, the reason being the problems were so simple, with uniform (homogeneous) stress and strain (as indeed also in the first example which follows). Whenever more complex problems are encountered, with non-uniform stress and strains, (3) and perhaps (2) need to be considered to solve for the stress and strain.

7.1.2 Structures with Uniform Members

A uniform axial member is one with cross-section A and modulus E constant along its length, and loaded with axial forces at its ends only.

Example

Consider the bar of initial length L shown in Fig. 7.1.4, subjected to equal and opposite end-forces F . The free-body (equilibrium) diagram of a section of the bar shown in Fig. 7.1.4b shows that the internal force is also F everywhere along the bar. The stress is thus everywhere $\sigma = F/A$ and the strain is everywhere

$$\varepsilon = \frac{F}{EA} \quad (7.1.4)$$

and, from Eq. 7.1.2, the bar extends in length by an amount

$$\Delta = \frac{FL}{EA} \quad (7.1.5)$$

Note that, although the force acting on the left-hand end is negative (acting in the $-x$ direction), the stress there is positive (see Fig. 7.1.3).

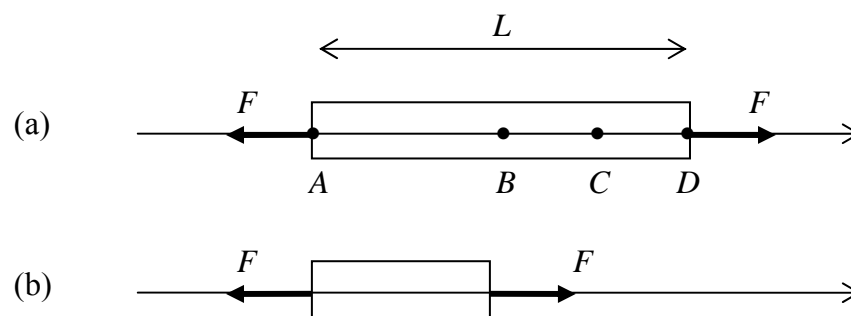


Figure 7.1.4: A uniform axial member; (a) subjected to axial forces F , (b) free-body diagram

Displacements need to be calculated relative to some datum displacement¹. For example, suppose that the displacement at the centre of the bar is zero, $u(B) = 0$, Fig. 7.1.4. Then, from Eqn. 7.1.3,

$$\begin{aligned} u(C) &= u(B) + \varepsilon(C - B) = \frac{F}{EA} \frac{L}{4} \\ u(D) &= u(B) + \varepsilon(D - B) = \frac{F}{EA} \frac{L}{2} \\ u(A) &= u(B) + \varepsilon(A - B) = -\frac{F}{EA} \frac{L}{2} \end{aligned} \quad (7.1.6)$$

■

¹ which is another way of saying that one can translate the bar left or right as a rigid body without affecting the stress or strain – but it does affect the displacements

Example

Consider the two-element structure shown in Fig. 7.1.5. The first element is built-in to a wall at end A , is of length L_1 , cross-sectional area A_1 and Young's modulus E_1 . The second element is attached at B and has properties L_2, A_2, E_2 . External loads F and P are applied at B and C as shown. An unknown reaction force R acts at the wall, at A . This can be determined from the force equilibrium equation for the complete structure:

$$R - F + P = 0 \quad (7.1.7)$$

Note that, as is usual, the reaction is assumed to act in the positive (x) direction. With R known, the stress $\sigma^{(1)}$ in the first element can be evaluated using the free-body diagram 7.1.5b, and $\sigma^{(2)}$ using Fig. 7.1.5c:

$$\sigma^{(1)} = \frac{P - F}{A_1}, \quad \sigma^{(2)} = \frac{P}{A_2} \quad (7.1.8)$$

and so the strain is

$$\varepsilon^{(1)} = \frac{P - F}{E_1 A_1}, \quad \varepsilon^{(2)} = \frac{P}{E_2 A_2} \quad (7.1.9)$$

Note that the stress and strain are *discontinuous* at B ².

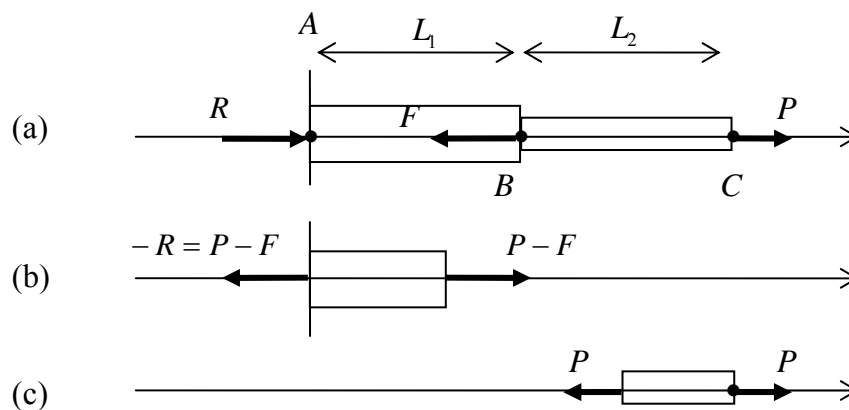


Figure 7.1.5: A two-element structure (a) subjected to axial forces F and P , (b,c) free-body diagrams

For each element, the total elongations Δ_i are

² this result, which can be viewed as a violation of equilibrium at B , is a result of the one-dimensional approximation of what is really a two-dimensional problem

$$\begin{aligned}\Delta_1 &= u(B) - u(A) = \frac{(P - F)L_1}{E_1 A_1} \\ \Delta_2 &= u(C) - u(B) = \frac{PL_2}{E_2 A_2}\end{aligned}\tag{7.1.10}$$

If $P > F$, then $\Delta_1 > 0$ as expected, with $R < 0$ and $\sigma > 0$.

Thus far, the stress and strain (and elongations) have been obtained. If one wants to evaluate the displacements, then one needs to ensure that the strains in each of the two elements are **compatible**, that is, that the elements fit together after deformation just like they did before deformation. In this example, the displacements at B and C are

$$u(B) = u(A) + \Delta_1, \quad u(C) = u(B) + \Delta_2\tag{7.1.11}$$

A **compatibility condition**, bringing together the separate relations in 7.1.11, is then

$$u(C) = u(A) + \frac{(P - F)L_1}{E_1 A_1} + \frac{PL_2}{E_2 A_2}\tag{7.1.12}$$

ensuring that $u(B)$ is unique. As in the previous example, the displacements can now be calculated if the displacement at any one (datum) point is known. Indeed, it is known that $u(A) = 0$.

■

Example

Consider next the similar situation shown in Fig. 7.1.6. Here, both ends of the two-element structure are built-in and there is only one applied force, F , at B . There are now two reaction forces, at ends A and C , but there is only one equilibrium equation to determine them:

$$R_A + F + R_C = 0\tag{7.1.13}$$

Any structure for which there are more unknowns than equations of equilibrium, so that the stresses cannot be determined without considering the deformation of the structure, is called a **statically indeterminate** structure³.

³ See the end of §2.3.3

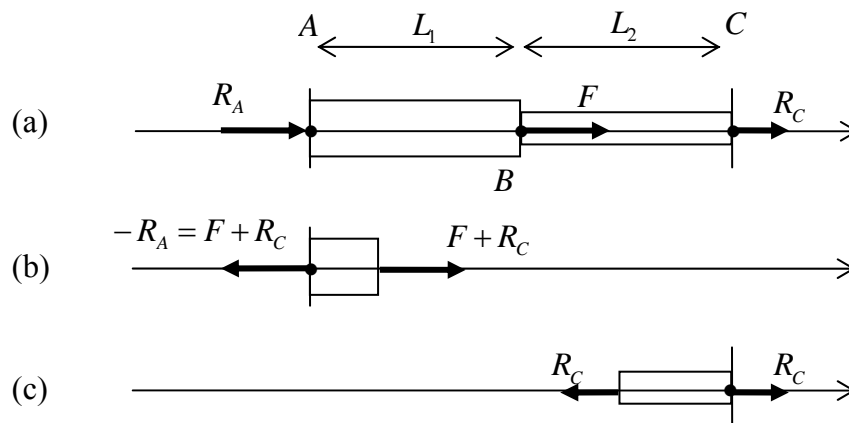


Figure 7.1.6: A two-element structure built-in and both ends; (a) subjected to an axial force F , (b,c) free-body diagrams

In terms of the unknown reactions, the strains are

$$\varepsilon^{(1)} = \frac{\sigma^{(1)}}{E_1} = -\frac{R_A}{E_1 A_1} = \frac{F + R_C}{E_1 A_1}, \quad \varepsilon^{(2)} = \frac{\sigma^{(2)}}{E_2} = \frac{R_C}{E_2 A_2} \quad (7.1.14)$$

and, for each element, the total elongations are

$$\Delta_1 = \frac{R_A L_1}{E_1 A_1}, \quad \Delta_2 = \frac{R_C L_2}{E_2 A_2} \quad (7.1.15)$$

Finally, compatibility of both elements implies that the total elongation $\Delta_1 + \Delta_2 = 0$. Using this relation with Eqn. 7.1.13-14 then gives

$$R_A = +F \frac{L_2 E_1 A_1}{L_1 E_2 A_2 + L_2 E_1 A_1}, \quad R_C = -F \frac{L_1 E_2 A_2}{L_1 E_2 A_2 + L_2 E_1 A_1} \quad (7.1.16)$$

The displacements can now be evaluated, for example,

$$u(B) = +F \frac{1}{E_1 A_1 / L_1 + E_2 A_2 / L_2} \quad (7.1.17)$$

so that a positive F displaces B to the right and a negative F displaces B to the left. ■

Note the general solution procedure in this last example, known as the **basic force method**:

Equilibrium + Compatibility of Strain in terms of unknown Forces
→ Solve equations for unknown Forces

The Stiffness Method

The **stiffness method** (also known as the **displacement method**) is a slight modification of the above solution procedure, where the final equations to be solved involve known forces and unknown displacements only:

Equilibrium in terms of Displacement
→ Solve equations for unknown Displacements

If one deals in displacements, one does not need to ensure compatibility (it will automatically be satisfied); compatibility only needs to be considered when dealing in strains (as in the previous example)⁴.

Example (The Stiffness Method)

Consider a series of three bars of cross-sectional areas A_1, A_2, A_3 , Young's moduli E_1, E_2, E_3 and lengths L_1, L_2, L_3 , Fig. 7.1.7. The first and third bars are built-in at points A and D , bars one and two meet at B and bars two and three meet at C . Forces P_B and P_C act at B and C respectively.

The force is constant in each bar, and for each bar there is a relation between the force F_i , and elongation, Δ_i , Eqn. 7.1.5:

$$F_i = k_i \Delta_i \quad \text{where} \quad k_i = \frac{A_i E_i}{L_i} \quad (7.1.18)$$

Here, k_i is the effective **stiffness** of each bar. The elongations are related to the displacements, $\Delta_1 = u_B - u_A$ etc., so that, with $u_A = u_D = 0$,

$$F_1 = k_1 u_B, \quad F_2 = k_2 (u_C - u_B), \quad F_3 = -k_3 u_C \quad (7.1.19)$$

There are two **degrees of freedom** in this problem, that is, two nodes are free to move. One therefore needs two equilibrium equations. One could use any two of

$$-F_1 + P_B + P_C + F_3 = 0, \quad -F_1 + P_B + F_2 = 0, \quad -F_2 + P_C + F_3 = 0 \quad (7.1.20)$$

In the stiffness method, one uses the second and third of these; the second is the “node B ” equation and the third is the “node C ” equation. Substituting Eqns. 7.1.19 into 7.1.20 leads to the system of two equations

$$\begin{aligned} -(k_1 + k_2)u_B + k_2 u_C &= -P_B \\ +k_2 u_B - (k_2 + k_3)u_C &= -P_C \end{aligned} \quad (7.1.21)$$

⁴ the reason is: if you know the displacements, you know where every particle is and you know the strains and everything else; if you *only know the strains*, you know the *change in displacement*, but you do not know the actual displacements. You need some extra information to know the displacements – this is the compatibility equation

which can be solved for the two unknown nodal displacements.

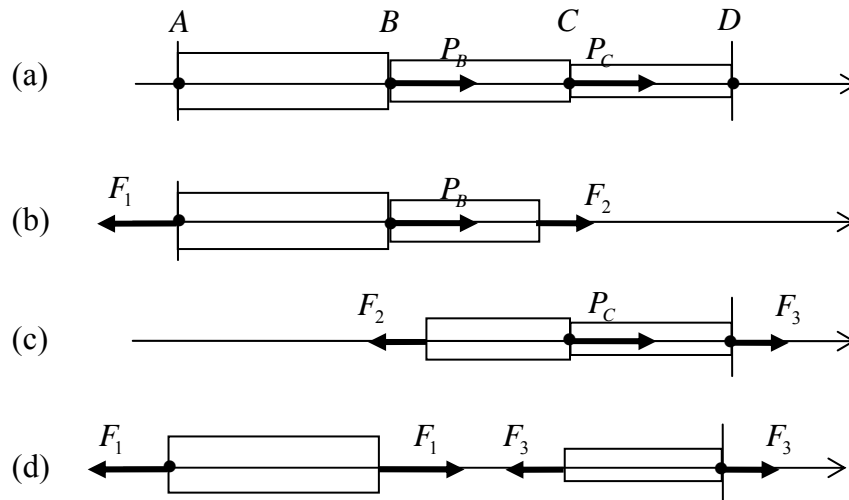


Figure 7.1.7: three bars in series; (a) subjected to external loads, (b,c,d) free-body diagrams

Equations 7.1.21 can also be written in the matrix form

$$\begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{bmatrix} \begin{bmatrix} u_B \\ u_C \end{bmatrix} = \begin{bmatrix} -P_B \\ -P_C \end{bmatrix} \quad (7.1.22)$$

Note that it was not necessary to evaluate the reactions to obtain a solution. Once the forces have been found, the reactions can be found using the free-body diagram of Fig. 7.1.7d.

The stiffness method is a very systematic procedure. It can be used to solve for structures with many elements, with the two equations 7.1.21, 7.1.22, replaced by a large system of equations which can be solved numerically using a computer.

7.1.3 Structures with Non-uniform Members

Consider the structure shown in Fig. 7.1.8, an axial bar consisting of two separate components bonded together. The components have Young's moduli E_1, E_2 and cross-sectional areas A_1, A_2 . The bar is subjected to equal and opposite forces F as shown, in such a way that axial deformations occur, that is, the cross-sections remain perpendicular to the x axis throughout the deformation.

Since there are only axial deformations, the strain is constant over a cross-section. However, the stress is not uniform, with $\sigma_1 = E_1 \varepsilon$ and $\sigma_2 = E_2 \varepsilon$; on any cross-section, the stress is higher in the stiffer component. The resultant force acting on each component is $F_1 = E_1 A_1 \varepsilon$ and $F_2 = E_2 A_2 \varepsilon$. Since $F_1 + F_2 = F$, the total elongation is

$$\Delta = \frac{FL}{E_1 A_1 + E_2 A_2} \quad (7.1.23)$$

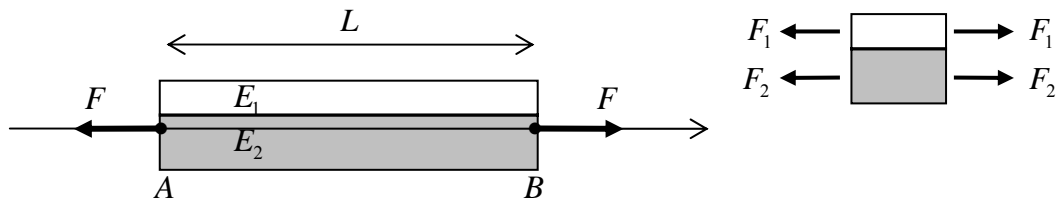


Figure 7.1.8: A bar consisting of two separate materials bonded together

7.1.4 Resultant Force and Moment

Consider the force and moments acting over any cross-section, Fig. 7.1.9. The resultant force is the integral of the stress times elemental area over the cross section, Eqn. 3.1.2,

$$F = \int_A \sigma dA \quad (7.1.24)$$

There are two moments; the moment M_y about the y axis and M_z about the z axis,

$$M_y = \int_A z \sigma dA, \quad M_z = -\int_A y \sigma dA \quad (7.1.25)$$

Positive moments are defined through the **right hand rule**, i.e. with the thumb of the right hand pointing in the positive y direction, the closing of the fingers indicates the positive M_y ; the negative sign in Eqn. 7.1.25b is due to the fact that a positive stress with $y > 0$ would lead to a negative moment M_z .

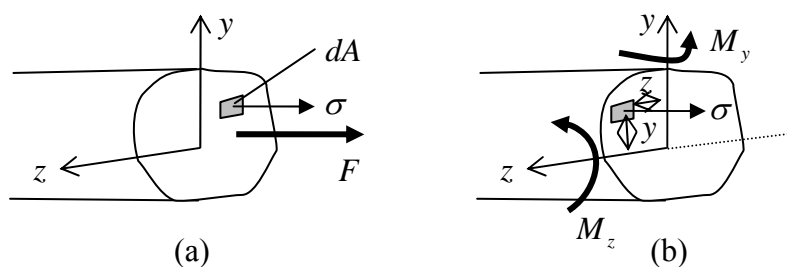


Figure 7.1.9: Resultants on a cross-section; (a) resultant force, (b) resultant moments

Consider now the case where *the stress is constant over a cross-section*. (Since it is assumed that the strain is constant over the cross-section, from Eqn. 7.1.1 this will occur when the Young's modulus is constant.) In that case, Eqns. 7.1.24-25 can be re-written as

$$F = \sigma A, \quad M_y = \sigma \int_A z dA, \quad M_z = -\sigma \int_A y dA \quad (7.1.25)$$

The quantities $\int_A z dA$ and $\int_A y dA$ are the **first moments of area** about, respectively, the y and z axes. These are equal to $\bar{z}A$ and $\bar{y}A$, where (\bar{y}, \bar{z}) are the coordinates of the centroid of the section (see Eqn. 3.2.2). Taking the x axis to run through the centroid, $\bar{y} = \bar{z} = 0$, which results in $M_y = M_z = 0$. Thus, a resultant axial force which acts through the centroid of the cross-section ensures that there is no moment/rotation of that cross-section, the main assumption of this section.

For the non-uniform member of Fig. 7.1.8, since the resultant of a constant stress over an area is a force acting through the centroid of that area, the forces F_1, F_2 act through the centroids of the respective areas A_1, A_2 . The precise location of the total resultant force F can be determined by taking the moments of the forces F_1, F_2 about the y and z axes, and equating this to the moment of the force F about these axes.

7.1.5 Problems

1. Consider the *rigid* beam supported by two deformable bars shown below. The bars have properties L_1, A_1 and L_2, A_2 and have the same Young's modulus E . They are separated by a distance L . The beam supports an arbitrary load at position x , as shown. What is x if the beam is to remain horizontal after deformation.

