6.3 Anisotropic Elasticity

There are many materials which, although well modelled using the linear elastic model, are not nearly isotropic. Examples are wood, composite materials and many biological materials. The mechanical properties of these materials differ in different directions. Materials with this direction dependence are called anisotropic (see Section 5.2.7).

6.3.1 Material Constants

The most general form of Hooke's law, the **generalised Hooke's Law**, for a linear elastic material is

$$\begin{bmatrix} \sigma_1 = \sigma_{xx} \\ \sigma_2 = \sigma_{yy} \\ \sigma_3 = \sigma_{zz} \\ \sigma_4 = \sigma_{yz} \\ \sigma_5 = \sigma_{xz} \\ \sigma_6 = \sigma_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 = \varepsilon_{xx} \\ \varepsilon_2 = \varepsilon_{yy} \\ \varepsilon_3 = \varepsilon_{zz} \\ \varepsilon_4 = \varepsilon_{yz} \\ \varepsilon_5 = \varepsilon_{xz} \\ \varepsilon_6 = \varepsilon_{xy} \end{bmatrix} \qquad (6.3.1)$$

where each stress component depends (linearly) on all strain components. This new notation, with only one subscript for the stress and strain, numbered from 1…6, is helpful as it allows the equations of anisotropic elasticity to be written in matrix form. The 36 C_{ij}'s are material constants called the **stiffnesses**, and in principle are to be obtained from experiment. The matrix of stiffnesses is called the **stiffness matrix**. Note that these equations imply that a normal stress σ_{xx} will induce a material element to not only stretch in the *x* direction and contract laterally, but to undergo shear strain too, as illustrated schematically in Fig. 6.3.1.

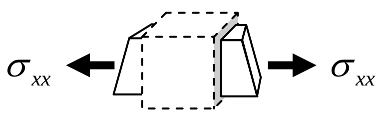


Figure 6.3.1: an element undergoing shear strain when subjected to a normal stress only

In section 8.4.3, when discussing the strain energy in an elastic material, it will be shown that it is necessary for the stiffness matrix to be *symmetric* and so there are only 21 independent elastic constants in the most general case of anisotropic elasticity.

Eqns. 6.3.1 can be inverted so that the strains are given explicitly in terms of the stresses:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad (6.3.2)$$

The S_{ij}'s here are called **compliances**, and the matrix of compliances is called the **compliance matrix**. The bottom half of the compliance matrix has been omitted since it too is symmetric.

It is difficult to model fully anisotropic materials due to the great number of elastic constants. Fortunately many materials which are not fully isotropic still have certain **material symmetries** which simplify the above equations. These material types are considered next.

6.3.2 Orthotropic Linear Elasticity

An **orthotropic** material is one which has three orthogonal planes of microstructural symmetry. An example is shown in Fig. 6.3.2a, which shows a glass-fibre composite material. The material consists of thousands of very slender, long, glass fibres bound together in bundles with oval cross-sections. These bundles are then surrounded by a plastic binder material. The continuum model of this composite material is shown in Fig. 6.3.2b wherein the fine microstructural details of the bundles and surrounding matrix are "smeared out" and averaged. Three mutually perpendicular planes of symmetry can be passed through each point in the continuum model. The x, y, z axes forming these planes are called the **material directions**.

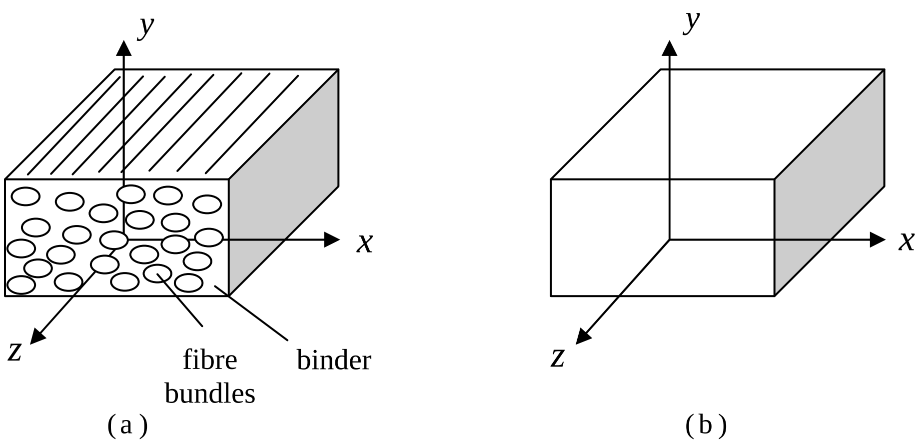


Figure 6.3.2: an orthotropic material; (a) microstructural detail, (b) continuum model

The material symmetry inherent in the orthotropic material reduces the number of independent elastic constants. To see this, consider an element of orthotropic material subjected to a shear strain $\varepsilon_6 \ (= \varepsilon_{xy})$ and also a strain $-\varepsilon_6 \ (= -\varepsilon_{xy})$, as in Fig. 6.3.3.

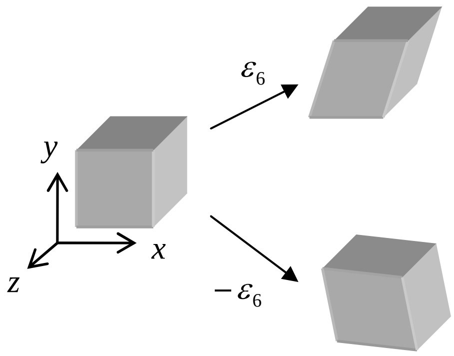


Figure 6.3.3: an element of orthotropic material undergoing shear strain

From Eqns. 6.3.1, the stresses induced by a strain ε_6 only are

$$\begin{aligned} \sigma_1 &= C_{16}\varepsilon_6, \quad \sigma_2 = C_{26}\varepsilon_6, \quad \sigma_3 = C_{36}\varepsilon_6 \\ \sigma_4 &= C_{46}\varepsilon_6, \quad \sigma_5 = C_{56}\varepsilon_6, \quad \sigma_6 = C_{66}\varepsilon_6 \end{aligned} \tag{6.3.3}$$

The stresses induced by a strain $-\varepsilon_6$ only are (the prime is added to distinguish these stresses from those of Eqn. 6.3.3)

$$\begin{aligned} \sigma_1' &= -C_{16}\varepsilon_6, \quad \sigma_2' = -C_{26}\varepsilon_6, \quad \sigma_3' = -C_{36}\varepsilon_6 \\ \sigma_4' &= -C_{46}\varepsilon_6, \quad \sigma_5' = -C_{56}\varepsilon_6, \quad \sigma_6' = -C_{66}\varepsilon_6 \end{aligned} \tag{6.3.4}$$

These stresses, together with the strain, are shown in Fig. 6.3.4 (the microstructure is also indicated)

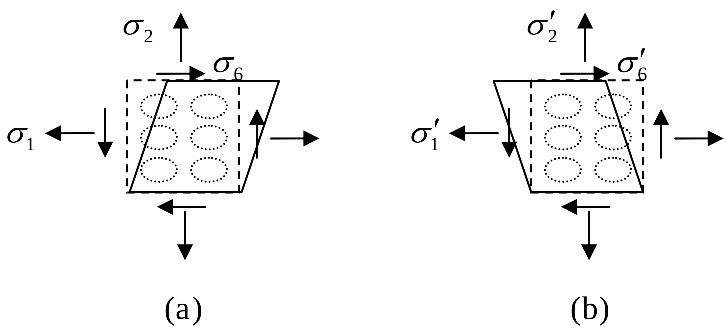


Figure 6.3.4: an element of orthotropic material undergoing shear strain; (a) positive strain, (b) negative strain

Because of the symmetry of the material (print this page out, turn it over, and Fig. 6.3.4a viewed from the "other side" of the page is the same as Fig. 6.3.4b on "this side" of the page), one would expect the normal stresses in Fig. 6.3.4 to be the same, $\sigma_1 = \sigma_1'$,

$\sigma_2 = \sigma_2'$, but the shear stresses to be of opposite sign, $\sigma_6 = -\sigma_6'$. Eqns. 6.3.3-4 then imply that

$$C_{16} = C_{26} = C_{36} = C_{46} = C_{56} = 0 \tag{6.3.5}$$

Similar conclusions follow from considering shear strains in the other two planes:

$$\begin{aligned} &\varepsilon_5 : C_{15} = C_{25} = C_{35} = C_{45} = 0 \\ &\varepsilon_4 : C_{14} = C_{24} = C_{34} = 0 \end{aligned} \tag{6.3.6}$$

The stiffness matrix is thus reduced, and there are only *nine* independent elastic constants:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{6.3.7}$$

These equations can be inverted to get, introducing elastic constants E, ν and G in place of the $S_{ij}'s$:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.8}$$

The nine independent constants here have the following meanings:

E_i is the Young's modulus (stiffness) of the material in direction $i = 1,2,3$; for example, $\sigma_1 = E_1\varepsilon_1$ for uniaxial tension in the direction 1.

ν_{ij} is the Poisson's ratio representing the ratio of a transverse strain to the applied strain in *uniaxial tension*; for example, $\nu_{12} = -\varepsilon_2 / \varepsilon_1$ for uniaxial tension in the direction 1.

G_{ij} are the shear moduli representing the shear stiffness in the corresponding plane; for example, G_{12} is the shear stiffness for shearing in the 1-2 plane.

If the 1-axis has long fibres along that direction, it is usual to call G_{12} and G_{13} the **axial shear moduli** and G_{23} the **transverse** (**out-of-plane**) **shear modulus**.

Note that, from symmetry of the stiffness matrix,

$$\nu_{23}E_3 = \nu_{32}E_2, \quad \nu_{13}E_3 = \nu_{31}E_1, \quad \nu_{12}E_2 = \nu_{21}E_1 \tag{6.3.9}$$

An important feature of the orthotropic material is that there is no **shear coupling** with respect to the material axes. In other words, normal stresses result in normal strains only and shear stresses result in shear strains only.

Note that there will in general be shear coupling when the reference axes used, x, y, z, are not aligned with the material directions $1, 2, 3$. For example, suppose that the $x-y$ axes were oriented to the material axes as shown in Fig. 6.3.5. Assuming that the material constants were known, the stresses and strains in the constitutive equations 6.3.8 can be transformed into $\varepsilon_{xx}, \varepsilon_{xy}$, etc. and σ_{xx}, σ_{xy}, etc. using the strain and stress transformation equations. The resulting matrix equations relating the strains $\varepsilon_{xx}, \varepsilon_{xy}$ to the stresses σ_{xx}, σ_{xy} will then not contain zero entries in the stiffness matrix, and normal stresses, e.g. σ_{xx}, will induce shear strain, e.g. ε_{xy}, and shear stress will induce normal strain.

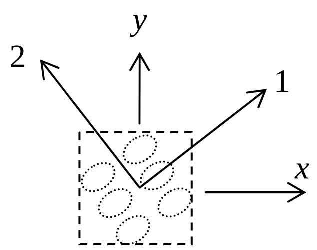


Figure 6.3.5: reference axes not aligned with the material directions

6.3.3 Transversely Isotropic Linear Elasticity

A **transversely isotropic** material is one which has a single material direction and whose response in the plane orthogonal to this direction is isotropic. An example is shown in Fig. 6.3.6, which again shows a glass-fibre composite material with aligned fibres, only now the cross-sectional shapes of the fibres are circular. The characteristic material direction is z and the material is isotropic in any plane parallel to the $x-y$ plane. The material properties are the same in all directions transverse to the fibre direction.

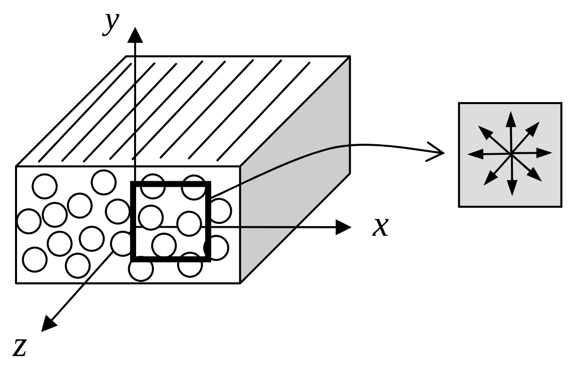


Figure 6.3.6: a transversely isotropic material

This extra symmetry over that inherent in the orthotropic material reduces the number of independent elastic constants further. To see this, consider an element of transversely isotropic material subjected to a normal strain $\varepsilon_1 \left(= \varepsilon_{xx}\right)$ only of magnitude ε, Fig. 6.3.7a, and also a normal strain $\varepsilon_2 \left(= \varepsilon_{yy}\right)$ of the same magnitude, ε, Fig. 6.3.7b. The $x - y$ plane is the plane of isotropy.

Figure 6.3.7: elements of a transversely isotropic material undergoing normal strain in the plane of isotropy

From Eqns. 6.3.7, the stresses induced by a strain $\varepsilon_1 = \varepsilon$ only are

$$\begin{gathered} \sigma_1 = C_{11}\varepsilon, \quad \sigma_2 = C_{21}\varepsilon, \quad \sigma_3 = C_{31}\varepsilon \\ \sigma_4 = 0, \quad \sigma_5 = 0, \quad \sigma_6 = 0 \end{gathered} \tag{6.3.10}$$

The stresses induced by the strain $\varepsilon_2 = \varepsilon$ only are (the prime is added to distinguish these stresses from those of Eqn. 6.3.10)

$$\begin{gathered} \sigma_1' = C_{12}\varepsilon, \quad \sigma_2' = C_{22}\varepsilon, \quad \sigma_3' = C_{32}\varepsilon \\ \sigma_4' = 0, \quad \sigma_5' = 0, \quad \sigma_6' = 0 \end{gathered} \tag{6.3.11}$$

Because of the isotropy, the $\sigma_1 (= \sigma_{xx})$ due to the ε_1 should be the same as the $\sigma_2 = (\sigma_{yy})$ due to the ε_2, and it follows that $C_{11} = C_{22}$. Further, the $\sigma_3 (= \sigma_{zz})$ should be the same for both, and so $C_{31} = C_{32}$.

Further simplifications arise from consideration of shear deformations, and rotations about the material axis, and one finds that $C_{44} = C_{55}$ and $C_{66} = C_{11} - C_{12}$.

The stiffness matrix is thus reduced, and there are only *five* independent elastic constants:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{6.3.12}$$

with '3' being the material direction. These equations can be inverted to get, introducing elastic constants E, ν and G in place of the S_{ij}'s. One again gets Eqn. 6.3.8, but now

$$E_1 = E_2, \quad \nu_{12} = \nu_{21}, \quad \nu_{13} = \nu_{23}, \quad \nu_{31} = \nu_{32}, \quad G_{13} = G_{23} \tag{6.3.13}$$

so

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.14}$$

with, due to symmetry,

$$\nu_{13} / E_1 = \nu_{31} / E_3 \tag{6.3.15}$$

Eqns. 6.3.13-15 seem to imply that there are 6 independent constants; however, the transverse modulus G_{12} is related to the transverse Poisson ratio and the transverse stiffness through (see Eqn. 6.1.5, and 6.3.20 below, for the isotropic version of this relation)

$$G_{12} = \frac{E_1}{2(1+\nu_{12})} \tag{6.3.16}$$

These equations are often expressed in terms of "*a*" for fibre (or "*a*" for axial) and "*t*" for transverse:

$$\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}=\begin{bmatrix}\dfrac{1}{E_t} & -\dfrac{\nu_t}{E_t} & -\dfrac{\nu_f}{E_f} & 0 & 0 & 0\\ -\dfrac{\nu_t}{E_t} & \dfrac{1}{E_t} & -\dfrac{\nu_f}{E_f} & 0 & 0 & 0\\ -\dfrac{\nu_f}{E_f} & -\dfrac{\nu_f}{E_f} & \dfrac{1}{E_f} & 0 & 0 & 0\\ 0 & 0 & 0 & \dfrac{1}{2G_f} & 0 & 0\\ 0 & 0 & 0 & 0 & \dfrac{1}{2G_f} & 0\\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{2G_t}\end{bmatrix}\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix} \qquad (6.3.17)$$

6.3.4 Isotropic Linear Elasticity

An isotropic material is one for which the material response is independent of orientation. The symmetry here further reduces the number of elastic constants to *two*, and the stiffness matrix reads

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ & C_{11} & C_{12} & 0 & 0 & 0\\ & & C_{11} & 0 & 0 & 0\\ & & & C_{11}-C_{12} & 0 & 0\\ & & & & C_{11}-C_{12} & 0\\ & & & & & C_{11}-C_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix} \qquad (6.3.18)$$

These equations can be inverted to get, introducing elastic constants E, ν and G,

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{6.3.19}$$

with

$$\frac{1}{2G} = \frac{1+\nu}{E} \tag{6.3.20}$$

which are Eqns. 6.1.8 and 6.1.5.

Eqns. 6.3.18 can also be written concisely in terms of the engineering constants E, ν and G with the help of the **Lamé constants**, λ and μ:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{6.3.21}$$

with

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \quad (=G) \tag{6.3.22}$$

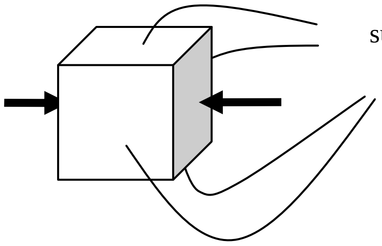
6.3.5 Problems

1. A piece of orthotropic material is loaded by a uniaxial stress σ_1 (aligned with the material direction '1'). What are the strains in the material, in terms of the engineering constants?

2. A specimen of bone in the shape of a cube is fixed and loaded by a compressive stress $\sigma = 1\text{MPa}$ as shown below. The bone can be considered to be orthotropic, with material properties

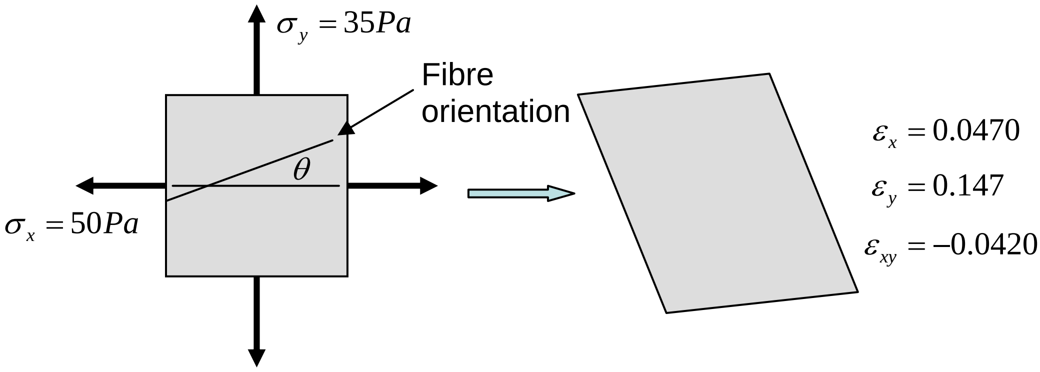
$$E_1 = 6.91\text{GPa}, \quad E_2 = 8.51\text{GPa}, \quad E_3 = 18.4\text{GPa}$$
$$G_{12} = 2.41\text{GPa}, \quad G_{13} = 3.56\text{GPa}, \quad G_{23} = 4.91\text{GPa}$$
$$\nu_{21} = 0.62, \quad \nu_{31} = 0.32, \quad \nu_{32} = 0.31$$

What are the stresses and strains which arise from the test according to this model (the bone is compressed along the '1' direction)?



3. Consider a block of transversely isotropic material subjected to a compressive stress $\sigma_1 = -p$ (perpendicular to the material direction) and constrained from moving in the other two perpendicular directions (as in Problem 2). Evaluate the stresses σ_2 and σ_3 in terms of the engineering constants E_t, E_f and ν_t, ν_f.

4. A strip of skin is tested in biaxial tension as shown below. The measured stresses and strains are as given in the figure. The orientation of the fibres in the material is later measured to be $\theta = 20^o$.



(a) Calculate the normal stresses along and transverse to the fibres, and the corresponding shear stress. (Hint: use the stress transformation equations.)

(b) Calculate the normal strains along and transverse to the fibres, and the corresponding shear strain. (Hint: use the strain transformation equations.)

(c) Assuming the material to be orthotropic, determine the elastic constants of the material (assume the stiffness in the fibre direction to be five times greater than the stiffness in the transverse direction). Note: because the material is thin, one can take $\sigma_3 = \sigma_4 = \sigma_5 = 0$.

(d) Calculate the magnitude and orientations of the principal normal stresses and strains. (Hint: the principal directions of stress are where there is zero shear stress.)

(e) Do the principal directions of stress and strain coincide?

5. A biaxial test is performed on a roughly planar section of skin (thickness 1mm) from the back of a test-animal. The test axes (*x* and *y*) are aligned such that deformation is induced in the skin along the spinal direction and transverse to this direction, under the assumption that the fibres are oriented principally in these directions. However, it is found during the experiment that shear stresses are necessary to maintain a biaxial deformation state. Measured stresses are

$$\sigma_{xx} = 5\text{kPa}, \quad \sigma_{yy} = 2\text{kPa}, \quad \sigma_{xy} = 1\text{kPa}$$

Determine the in-plane orientation of the fibres given the data $E_1 = 1000\,\text{kPa}$, $E_2 = 500\,\text{kPa}$, $G_6 = 500\,\text{kPa}$, $\nu_{21} = 0.2$.

[Hint: derive an expression for ε_{xy} involving θ only, where θ is the inclination of the material axes from the $x - y$ axes]