6.1 The Linear Elastic Model

6.1.1 The Linear Elastic Model

Repeating some of what was said in Section 5.3: the Linear Elastic model is used to describe materials which respond as follows:

(i) the strains in the material are small[1] (**linear**)
(ii) the stress is proportional to the strain, $\sigma \propto \varepsilon$ (**linear**)
(iii) the material returns to its original shape when the loads are removed, and the unloading path is the same as the loading path (**elastic**)
(iv) there is no dependence on the rate of loading or straining (**elastic**)

From the discussion in the previous chapter, this model well represents the engineering materials up to their elastic limit. It also models well almost any material provided the stresses are sufficiently small.

The stress-strain (loading and unloading) curve for the Linear Elastic solid is shown in Fig. 6.1.1a. Other possible responses are shown in Figs. 6.1.1b,c. Fig. 6.1.1b shows the typical response of a rubbery-type material and many biological tissues; these are **non-linear elastic** materials. Fig. 6.1.1c shows the typical response of **viscoelastic** materials (see Chapter 10) and that of many plastically and viscoplastically deforming materials (see Chapters 11 and 12).

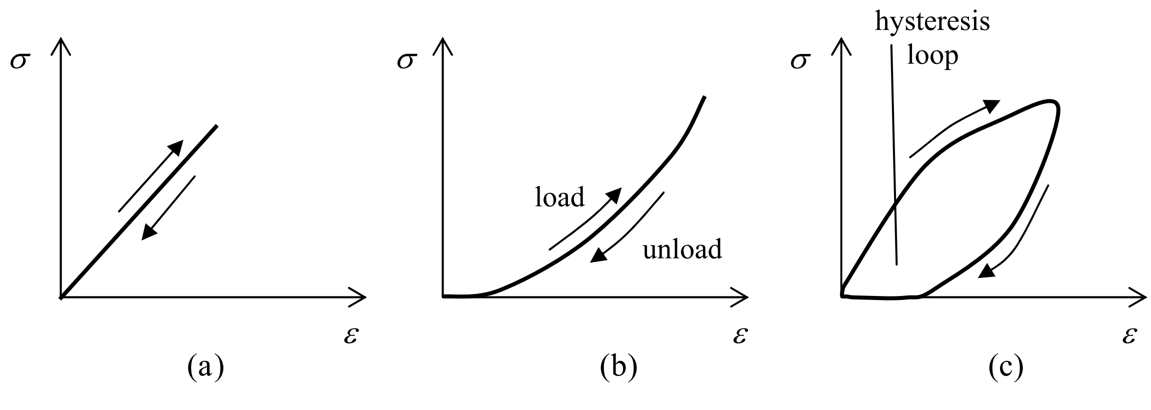

Figure 6.1.1: Different stress-strain relationships; (a) linear elastic, (b) non-linear elastic, (c) viscoelastic/plastic/viscoplastic

It will be assumed at first that the material is isotropic and homogeneous. The case of an anisotropic elastic material is discussed in Section 6.3.

[1] if the small-strain approximation is not made, the stress-strain relationship will be inherently non-linear; the actual strain, Eqn. 4.1.7, involves (non-linear) squares and square-roots of lengths

6.1.2 Stress-Strain Law

Consider a cube of material subjected to a uniaxial tensile stress σ_{xx}, Fig. 6.1.2a. One would expect it to respond by extending in the *x* direction, $\varepsilon_{xx} > 0$, and to contract laterally, so $\varepsilon_{yy} = \varepsilon_{zz} < 0$, these last two being equal because of the isotropy of the material. With stress proportional to strain, one can write

$$\varepsilon_{xx} = \frac{1}{E}\sigma_{xx}, \quad \varepsilon_{yy} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{xx} \tag{6.1.1}$$

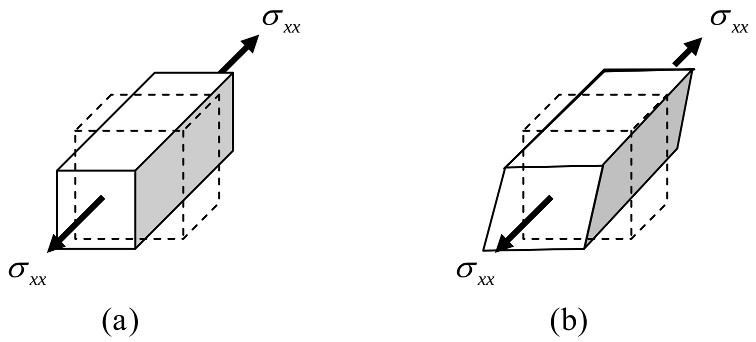

Figure 6.1.2: an element of material subjected to a uniaxial stress; (a) normal strain, (b) shear strain

The constant of proportionality between the normal stress and strain is the Young's Modulus, Eqn. 5.2.5, the measure of the stiffness of the material. The material parameter ν is the Poisson's ratio, Eqn. 5.2.6. Since $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$, it is a measure of the contraction relative to the normal extension.

Because of the isotropy/symmetry of the material, the shear strains are zero, and so the deformation of Fig. 6.1.2b, which shows a non-zero ε_{xy}, is not possible – shear strain can arise if the material is not isotropic.

One can write down similar expressions for the strains which result from a uniaxial tensile σ_{yy} stress and a uniaxial σ_{zz} stress:

$$\begin{aligned} \varepsilon_{yy} &= \frac{1}{E}\sigma_{yy}, \quad \varepsilon_{xx} = \varepsilon_{zz} = -\frac{\nu}{E}\sigma_{yy} \\ \varepsilon_{zz} &= \frac{1}{E}\sigma_{zz}, \quad \varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu}{E}\sigma_{zz} \end{aligned} \tag{6.1.2}$$

Similar arguments can be used to write down the shear strains which result from the application of a shear stress:

$$\varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz} \tag{6.1.3}$$

The constant of proportionality here is the Shear Modulus μ, Eqn. 5.2.8, the measure of the resistance to shear deformation (the letter *G* was used in Eqn. 5.2.8 – both *G* and μ are used to denote the Shear Modulus, the latter in more "mathematical" and "advanced" discussions) .

The strain which results from a combination of all six stresses is simply the sum of the strains which result from each[2]:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right], \quad \varepsilon_{xy} = \frac{1}{2\mu}\sigma_{xy}, \quad \varepsilon_{xz} = \frac{1}{2\mu}\sigma_{xz}, \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right], \quad \varepsilon_{yz} = \frac{1}{2\mu}\sigma_{yz} \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]
\end{aligned} \tag{6.1.4}$$

These equations involve three material parameters. It will be proved in §6.3 that an isotropic linear elastic material can have only *two* independent material parameters and that, in fact,

$$\mu = \frac{E}{2(1+\nu)}. \tag{6.1.5}$$

This relation will be verified in the following example.

Example: Verification of Eqn. 6.1.5

Consider the simple shear deformation shown in Fig. 6.1.3, with $\varepsilon_{xy} > 0$ and all other strains zero. With the material linear elastic, the only non-zero stress is $\sigma_{xy} = 2\mu\varepsilon_{xy}$.

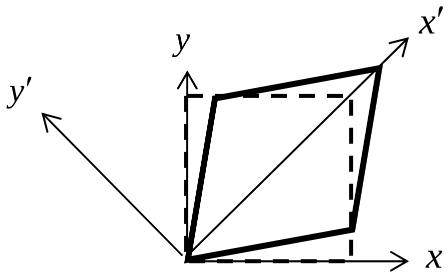

Figure 6.1.3: a simple shear deformation

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[2] this is called the **principle of linear superposition**: the "effect" of a sum of "causes" is equal to the sum of the individual "effects" of each "cause". For a linear relation, e.g. $\sigma = E\varepsilon$, the effects of two causes σ_1, σ_2 are $E\varepsilon_1$ and $E\varepsilon_2$, and the effect of the sum of the causes $\sigma_1 + \sigma_2$ is indeed equal to the sum of the individual effects: $E(\varepsilon_1 + \varepsilon_2) = E\varepsilon_1 + E\varepsilon_2$. This is not true of a non-linear relation, e.g. $\sigma = E\varepsilon^2$, since $E(\varepsilon_1 + \varepsilon_2)^2 \neq E\varepsilon_1^2 + E\varepsilon_2^2$

Using the strain transformation equations, Eqns. 4.2.2, the only non-zero strains in a second coordinate system $x' - y'$, with x' at $\theta = 45^{\circ}$ from the x axis (see Fig. 6.1.3), are $\varepsilon'_{xx} = +\varepsilon_{xy}$ and $\varepsilon'_{yy} = -\varepsilon_{xy}$. Because the material is isotropic, Eqns 6.1.4 hold also in this second coordinate system and so the stresses in the new coordinate system can be determined by solving the equations

$$
\begin{aligned}
\varepsilon'_{xx} &= +\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{xx} - \nu\left(\sigma'_{yy} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{xy} = 0 = \frac{1}{2\mu}\sigma'_{xy}, \quad \varepsilon'_{xz} = 0 = \frac{1}{2\mu}\sigma'_{xz}, \\
\varepsilon'_{yy} &= -\varepsilon_{xy} = \frac{1}{E}\left[\sigma'_{yy} - \nu\left(\sigma'_{xx} + \sigma'_{zz}\right)\right], \quad \varepsilon'_{yz} = 0 = \frac{1}{2\mu}\sigma'_{yz} \\
\varepsilon'_{zz} &= 0 \quad = \frac{1}{E}\left[\sigma'_{zz} - \nu\left(\sigma'_{xx} + \sigma'_{yy}\right)\right]
\end{aligned}
\tag{6.1.6}
$$

which results in

$$
\sigma'_{xx} = +\frac{E}{1+\nu}\varepsilon_{xy}, \quad \sigma'_{yy} = -\frac{E}{1+\nu}\varepsilon_{xy} \tag{6.1.7}
$$

But the stress transformation equations, Eqns. 3.4.8, with $\sigma_{xy} = 2\mu\varepsilon_{xy}$, give $\sigma'_{xx} = +2\mu\varepsilon_{xy}$ and $\sigma'_{yy} = -2\mu\varepsilon_{xy}$ and so Eqn. 6.1.5 is verified.

■

Relation 6.1.5 allows the Linear Elastic Solid stress-strain law, Eqn. 6.1.4, to be written as

$$
\boxed{
\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1+\nu}{E}\sigma_{xz} \\
\varepsilon_{yz} &= \frac{1+\nu}{E}\sigma_{yz}
\end{aligned}
}
\qquad \textbf{Stress-Strain Relations} \tag{6.1.8}
$$

This is known as **Hooke's Law**. These equations can be solved for the stresses to get

$$\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\left(\varepsilon_{yy} + \varepsilon_{zz}\right)\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\left(\varepsilon_{xx} + \varepsilon_{zz}\right)\right] \\
\sigma_{zz} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{zz} + \nu\left(\varepsilon_{xx} + \varepsilon_{yy}\right)\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy} \\
\sigma_{xz} &= \frac{E}{1+\nu}\varepsilon_{xz} \\
\sigma_{yz} &= \frac{E}{1+\nu}\varepsilon_{yz}
\end{aligned}$$

Stress-Strain Relations (6.1.9)

Values of E and ν for a number of materials are given in Table 6.1.1 below (see also Table 5.2.2).

Material	*E* (GPa)	ν
Grey Cast Iron	100	0.29
A316 Stainless Steel	196	0.3
A5 Aluminium	68	0.33
Bronze	130	0.34
Plexiglass	2.9	0.4
Rubber	0.001-2	0.4-0.49
Concrete	23-30	0.2
Granite	53-60	0.27
Wood (pinewood) fibre direction transverse direction	 17 1	 0.45 0.79

Table 6.1.1: Young's Modulus *E* and Poisson's Ratio ν for a selection of materials at 20^{o}C

Volume Change

Recall that the volume change in a material undergoing small strains is given by the sum of the normal strains (see Section 4.3). From Hooke's law, normal stresses cause normal strain and shear stresses cause shear strain. It follows that *normal stresses produce volume changes* and *shear stresses produce distortion* (change in shape), but no volume change.

6.1.3 Two Dimensional Elasticity

The above three-dimensional stress-strain relations reduce in the case of a two-dimensional stress state or a two-dimensional strain state.

Plane Stress

In plane stress (see Section 3.5), $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$, Fig. 6.1.5, so the stress-strain relations reduce to

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\sigma_{xx} &= \frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{1-\nu^2}\left[\nu\varepsilon_{xx} + \varepsilon_{yy}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}}$$

Stress-Strain Relations (Plane Stress) (6.1.10)

with

$$\begin{aligned}
&\varepsilon_{zz} = -\frac{\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right], \quad \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
&\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0
\end{aligned} \tag{6.1.11}$$

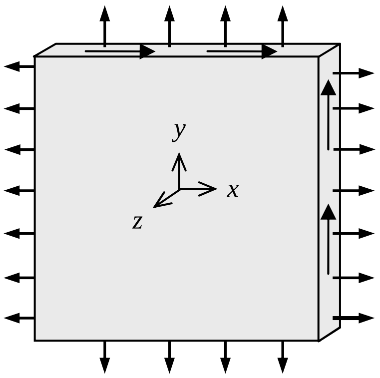


Figure 6.1.5: Plane stress

Note that the ε_{zz} strain is *not* zero. Physically, ε_{zz} corresponds to a change in thickness of the material perpendicular to the direction of loading.

Plane Strain

In plane strain (see Section 4.2), $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$, Fig. 6.1.6, and the stress-strain relations reduce to

$$\boxed{\begin{aligned}
\varepsilon_{xx} &= \frac{1+\nu}{E}\left[(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right] \\
\varepsilon_{yy} &= \frac{1+\nu}{E}\left[-\nu\sigma_{xx} + (1-\nu)\sigma_{yy}\right] \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \\
\\
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}\right] \\
\sigma_{xy} &= \frac{E}{1+\nu}\varepsilon_{xy}
\end{aligned}}$$

Stress-Strain Relations (Plane Strain) (6.1.12)

with

$$\begin{gathered}
\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0 \\
\sigma_{zz} = \nu\left[\sigma_{xx} + \sigma_{yy}\right], \quad \sigma_{xz} = \sigma_{yz} = 0
\end{gathered} \tag{6.1.13}$$

Again, note here that the stress component σ_{zz} is *not* zero. Physically, this stress corresponds to the forces preventing movement in the *z* direction.

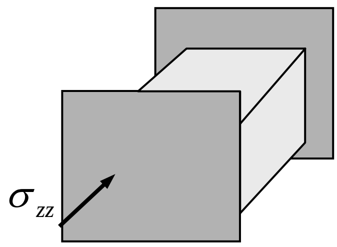


Figure 6.1.6 Plane strain - a thick component constrained in one direction

Similar Solutions

The expressions for plane stress and plane strain are very similar. For example, the plane strain constitutive law 6.1.12 can be derived from the corresponding plane stress expressions 6.1.10 by making the substitutions

$$E = \frac{E'}{1-\nu'^2}, \quad \nu = \frac{\nu'}{1-\nu'} \tag{6.1.14}$$

in 6.1.10 and then dropping the primes. The plane stress expressions can be derived from the plane strain expressions by making the substitutions

$$E = E'\frac{1+2\nu'}{(1+\nu')^2}, \quad \nu = \frac{\nu'}{1+\nu'} \tag{6.1.15}$$

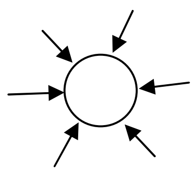
in 6.1.12 and then dropping the primes. Thus, if one solves a plane stress problem, one has automatically solved the corresponding plane strain problem, and *vice versa*.

6.1.4 Problems

1. A strain gauge at a certain point on the surface of a thin A5 Aluminium component (loaded in-plane) records strains of $\varepsilon_{xx} = 60\,\mu\text{m}$, $\varepsilon_{yy} = 30\,\mu\text{m}$, $\varepsilon_{xy} = 15\,\mu\text{m}$. Determine the principal stresses. (See Table 6.1.1 for the material properties.)

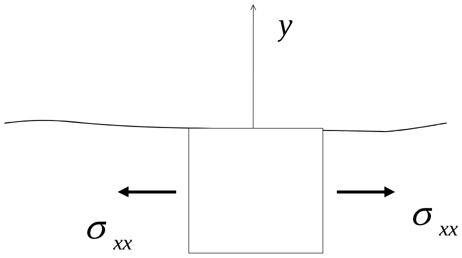
2. Use the stress-strain relations to prove that, for a linear elastic solid,
$$\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}} = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}$$
and, indeed,
$$\frac{2\sigma_{xz}}{\sigma_{xx} - \sigma_{zz}} = \frac{2\varepsilon_{xz}}{\varepsilon_{xx} - \varepsilon_{zz}}, \quad \frac{2\sigma_{yz}}{\sigma_{yy} - \sigma_{zz}} = \frac{2\varepsilon_{yz}}{\varepsilon_{yy} - \varepsilon_{zz}}$$
Note: from Eqns. 3.5.4 and 4.2.4, these show that the principal axes of stress and strain coincide for an *isotropic* elastic material

3. Consider the case of hydrostatic pressure in a linearly elastic solid:
$$\left[\sigma_{ij}\right] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$
as might occur, for example, when a spherical component is surrounded by a fluid under high pressure, as illustrated in the figure below. Show that the volumetric strain (Eqn. 4.3.5) is equal to
$$-p\frac{3(1-2\nu)}{E},$$
so that the Bulk Modulus, Eqn. 5.2.9, is
$$K = \frac{E}{3(1-2\nu)}$$



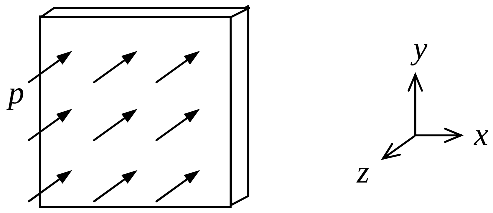
4. Consider again Problem 2 from §3.5.7.
   (a) Assuming the material to be linearly elastic, what are the strains? Draw a second material element (superimposed on the one shown below) to show the deformed shape of the square element – assume the displacement of the box-centre to be zero and that there is no rotation. Note how the free surface moves, even though there is no stress acting on it.

(b) What are the principal strains ε_1 and ε_2? You will see that the principal directions of stress and strain coincide (see Problem 2) – the largest normal stress and strain occur in the same direction.



5. Consider a very thin sheet of material subjected to a normal pressure *p* on one of its large surfaces. It is fixed along its edges. This is an example of a **plate** problem, an important branch of elasticity with applications to boat hulls, aircraft fuselage, etc.
   (a) write out the complete three dimensional stress-strain relations (both cases, strain in terms of stress, Eqns. 6.1.8, stress in terms of strain, Eqns. 6.1.9). Following the discussion on thin plates in section 3.5.4, the shear stresses σ_{xz}, σ_{yz}, can be taken to be zero throughout the plate. Simplify the relations using this fact, the pressure boundary condition on the large face and the coordinate system shown.
   (b) assuming that the through thickness change in the sheet can be neglected, show that

$$p = -\nu(\sigma_{xx} + \sigma_{yy})$$



6. A thin linear elastic rectangular plate with width *a* and height *b* is subjected to a uniform compressive stress σ_0 as shown below. Show that the slope of the plate diagonal shown after deformation is given by

$$\tan(\beta + \delta\beta) = \frac{b}{a}\left(\frac{1 + \nu\sigma_0 / E}{1 - \sigma_0 / E}\right)$$

What is the magnitude of $\delta\beta$ for a steel plate ($E = 210\text{GPa}$, $\nu = 0.3$) of dimensions $20 \times 20\ \text{cm}^2$ with $\sigma_0 = 1\text{MPa}$?