5 Material Behaviour and Mechanics Modelling

In this Chapter, the real physical response of various types of material to different types of loading conditions is examined. The means by which a mathematical model can be developed which can predict such real responses is also considered.

5.1 Mechanics Modelling

5.1.1 The Mechanics Problem

Typical questions which mechanics attempts to answer were given in Section 1.1. In the examples given, one invariably knows (some of) the forces (or stresses) acting on the material under study, be it due to the wind, water pressure, the weight of the human body, a moving train, and so on. One also often knows something about the displacements along some portion of the material, for example it might be fixed to the ground and so the displacements there are zero. A schematic of such a generic material is shown in Fig. 5.1.1 below.

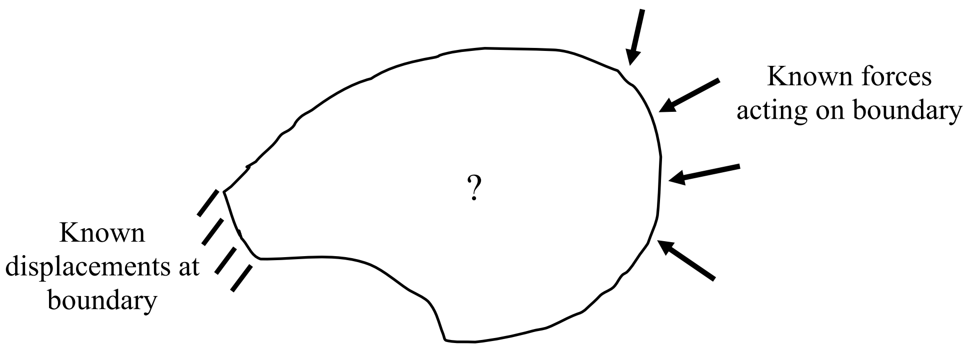

Figure 5.1.1: a material component; force and displacement are known along some portion of the boundary

The basic problem of mechanics is to determine what is happening *inside* the material. This means: what are the stresses and strains inside the material? With this information, one can answer further questions: Where are the stresses high? Where will the material first fail? What can we change to make the material function better? Where will the component move to? What is going on inside the material, at the microscopic level? Generally speaking, what is happening and what will happen?

One can relate the loads on the component to the stresses inside the body using equilibrium equations and one can relate the displacement to internal strains using kinematics relations. For example, consider again the simple rod subjected to tension forces examined in Section 3.3.1, shown again in Fig. 5.1.2. The internal normal stress σ_N on any plane oriented at an angle θ to the rod cross-section is related to the external force F through the equilibrium equation 3.3.1: $\sigma_N = F\cos^2\theta / A$, where A is the cross-sectional area. Similarly, if the ends undergo a separation/displacement of $\Delta = l - l_0$, Fig. 5.1.2b, the strain of any internal line element, at orientation θ, is $\varepsilon_N = \Delta\cos^2\theta / l_0$. However, there is no relationship between this internal stress and internal strain: for any given force, there is no way to determine the internal strain (and hence displacement of the rod); for any given displacement of the rod, there is no way to determine the internal stress (and hence force applied to the rod). The required relationship between stress and strain is discussed next.

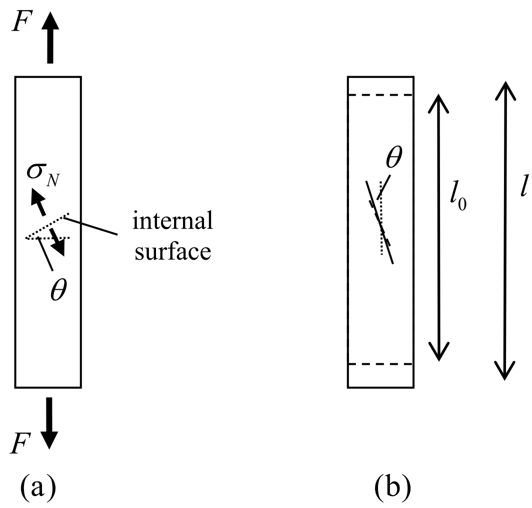

Figure 5.1.2: a slender rod; (a) internal stress due to external force, (b) internal strain due to gross displacement of rod (dotted = before straining)

5.1.2 Constitutive Equation

Stress was discussed in Chapter 3 and strain in Chapter 4. In all that discussion, no mention was made of the particular material under study, be it metallic, polymeric, biological or foodstuff (apart from the necessity that the strain be small when using the engineering strain). The concept of stress and the resulting theory of stress transformation, principal stresses and so on, are based on physical principles (Newton's Laws), which apply to *all* materials. The concept of strain is based, essentially, on geometry and trigonometry; again, it applies to all materials. However, it is the relationship *between* stress and strain which differs from material to material.

The relationship between the stress and strain for any particular material will depend on the microstructure of that material – what constitutes that material. For this reason, the stress-strain relationship is called the **constitutive relation**, or **constitutive law**. For example, metals consist of a closely packed lattice of atoms, whereas a rubber consists of a tangled mass of long-chain polymer molecules; for this reason, the strain in a metal will be different to that in rubber, when they are subjected to the same stress.

The constitutive equation allows the mechanics problem to be solved – this is shown schematically in Fig. 5.1.3.

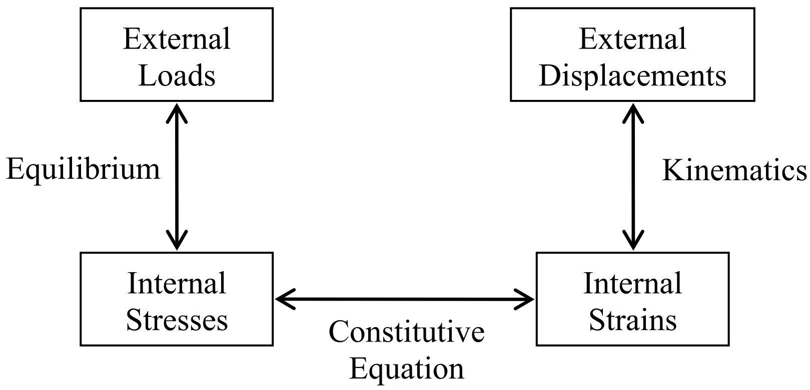

Figure 5.1.3: the role of the constitutive equation in the equations of mechanics

Example Constitutive Equations

A constitutive equation will be of the general form

$$\sigma = f(\varepsilon). \tag{5.1.1}$$

The simplest constitutive equation is a **linear elastic** relation, in which the stress is proportional to the strain:

$$\sigma \propto \varepsilon. \tag{5.1.2}$$

Although no real material satisfies precisely Eqn. 5.1.2, many do so approximately – this type of relation will be discussed in Chapters 6-8. More complex relations can involve the *rate* at which a material is strained or stressed; these types of relation will be discussed in Chapter 10.

More on constitutive equations will follow in Section 5.3.

5.1.3 Mathematical Model

Some of the questions asked earlier can be answered using experimentation. For example, one could use a car-crash test to determine the weakest points in a car. However, one cannot carry out multiple tests for each and every possible scenario – different car speeds, different obstacles into which it crashes, and so on; it would be too time-consuming and too expensive. The only practical way in which these questions can be answered is to develop a **mathematical model**. This model consists of the various equations of equilibrium and the kinematics, the constitutive relation, equations describing the shape of the material, etc. (see Fig. 5.1.3). The mathematical model will have many approximations to reality associated with it. For example, it might be assumed that the material is in the shape of a perfect sphere, when in fact it only resembles a sphere. It may be assumed that a load is applied at a "point" when in fact it is applied over a region of the material's surface. Another approximation in the mathematical model is the constitutive equation itself; the relation between stress and strain in any material can be extremely complex, and the constitutive equation can only be an approximation of the reality.

Once the mathematical model has been developed, the various equations can be solved and the model can then be used to *make a prediction*. The prediction of the model can now be tested against reality: a set of well-defined experiments can be carried out – does the material really move to where the model says it will move?

Simple models (simple constitutive relations) should be used as a first step. If the predictions of the model are wildly incorrect, the model can be adjusted (made more complicated), and the output tested again.

The equations associated with simple models can often be solved analytically, i.e. using a pen and paper. More complex models result in complex sets of equations which can only be solved approximately (though, hopefully, accurately) using a computer.

5.2 The Response of Real Materials

The constitutive equation was introduced in the previous section. The means by which the constitutive equation is determined is by carrying out experimental tests on the material in question. This topic is discussed in what follows.

5.2.1 The Tension Test

Consider the following key experiment, the **tensile test**, in which a small, usually cylindrical, specimen is gripped and stretched, usually at some given rate of stretching. A typical specimen would have diameter about 1cm and length 5cm, and larger ends so that it can be easily gripped, Fig. 5.2.1a. Specialised machines are used, for example the Instron testing machine shown in Fig. 5.2.1b.

1

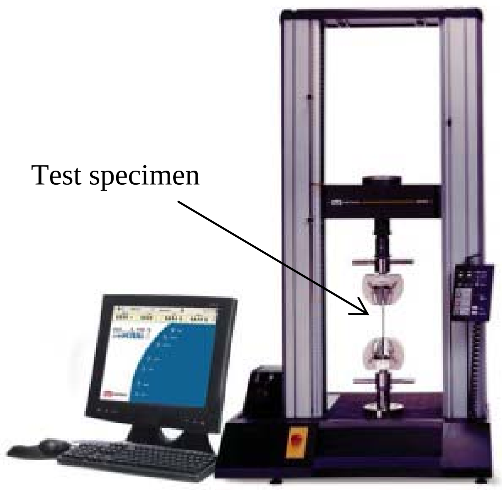

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Figure 5.2.1: the tension test; (a) test specimen, (b) testing machine

As the specimen is stretched, the force required to hold the specimen at a given displacement/stretch is recorded[1].

The Engineering Materials

For many of the (hard) engineering materials, the force/displacement curve will look something like that shown in Fig. 5.2.2. It will be found that the force is initially proportional to displacement as with the linear portion OA in Fig. 5.2.2. The following observations will also be made:

(1) if the load has not reached point A, and the material is then unloaded, the force/displacement curve will trace back along the line OA down to zero force and zero displacement; further loading and unloading will again be up and down OA.

(2) the loading curve remains linear up to a certain force level, the **elastic limit** of the material (point A). Beyond this point, **permanent deformations** are induced[2]; on

[1] the very precise details of how the test should be carried out are contained in the special standards for materials testing developed by the American Society for Testing and Materials (ASTM)

unloading to zero force (from point *B* to *C*), the specimen will have a permanent elongation. An example of this response (although not a tension test) can be seen with a paper clip – gently bend the clip and it will "spring back" (this is the *OA* behaviour); bend the clip too much (*AB*) and it will stay bent after you let go (*C*).

(3) above the elastic limit (from *A* to *B*), the material **hardens**, that is, the force required to maintain further stretching, unsurprisingly, keeps increasing. (However, some materials can **soften**, for example granular materials such as soils).

(4) the rate (speed) at which the specimen is stretched makes little difference to the results observed (at least if the speed and/or temperature is not too high).

(5) the strains up to the elastic limit are small, less than 1% (see below for more on strains).

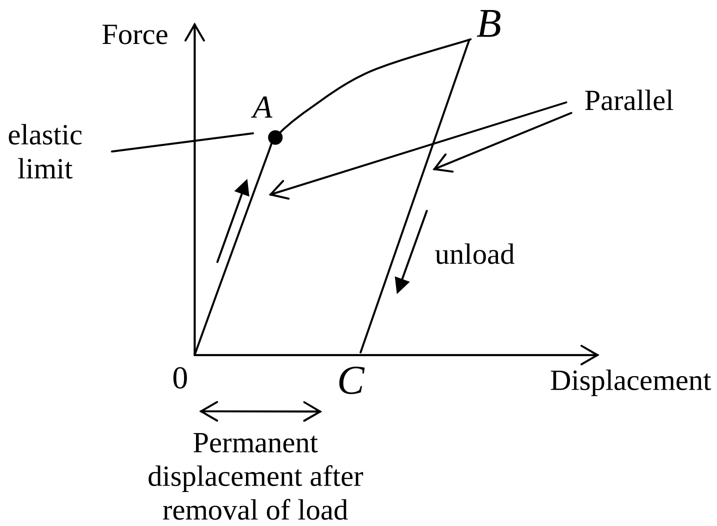

Figure 5.2.2: force/displacement curve for the tension test; typical response for engineering materials

Stress-Strain Curve

There are two definitions of stress used to describe the tension test. First, there is the force divided by the *original* cross sectional area of the specimen A_0; this is the **nominal stress** or **engineering stress**,

$$\sigma_n = \frac{F}{A_0} \tag{5.2.1}$$

Alternatively, one can evaluate the force divided by the (smaller) *current* cross-sectional area *A*, leading to the **true stress**

[2] if the tension tests are carried out extremely carefully, one might be able to distinguish between a point where the stress-strain curve ceases to be linear (the **proportional limit**) and the elastic limit (which will occur at a slightly higher stress)

$$\sigma = \frac{F}{A} \tag{5.2.2}$$

in which *F* and *A* are both changing with time. For small elongations, within the linear range OA, the cross-sectional area of the material undergoes negligible change and both definitions of stress are more or less equivalent.

Similarly, one can describe the deformation in two alternative ways. As discussed in Section 4.1.1, one can use the engineering strain

$$\varepsilon = \frac{l - l_0}{l_0} \tag{5.2.3}$$

or the true strain

$$\varepsilon_t = \ln\left(\frac{l}{l_0}\right) \tag{5.2.4}$$

Here, l_0 is the original specimen length and *l* is the current length. Again, at small deformations, the difference between these two strain measures is negligible (see Tabel 4.1).

The stress-strain diagram for a tension test can now be described using the true stress/strain or nominal stress/strain definitions, as in Fig. 5.2.3. The shape of the nominal stress/strain diagram, Fig. 5.2.3a, is of course the same as the graph of force versus displacement. *C* here denotes the point at which the maximum force the specimen can withstand has been reached. The nominal stress at *C* is called the **Ultimate Tensile Strength** (UTS) of the material.

After the UTS is reached, the specimen "necks", that is, the specimen begins to deform locally – with a very rapid reduction in cross-sectional area somewhere about the centre of the specimen until the specimen breaks, as indicated by the asterisk in Fig. 5.2.3. The appearance of a test specimen at each of these stages of the stress-strain curve is shown top of Fig. 5.2.3a.

For many materials, it will be observed that there is very little volume change during the permanent deformation phase, so $A_0 l_0 \approx Al$ and $\sigma = \sigma_N (1 + \varepsilon)$. This nominal stress to true stress conversion formula will only be valid up to the point of necking.

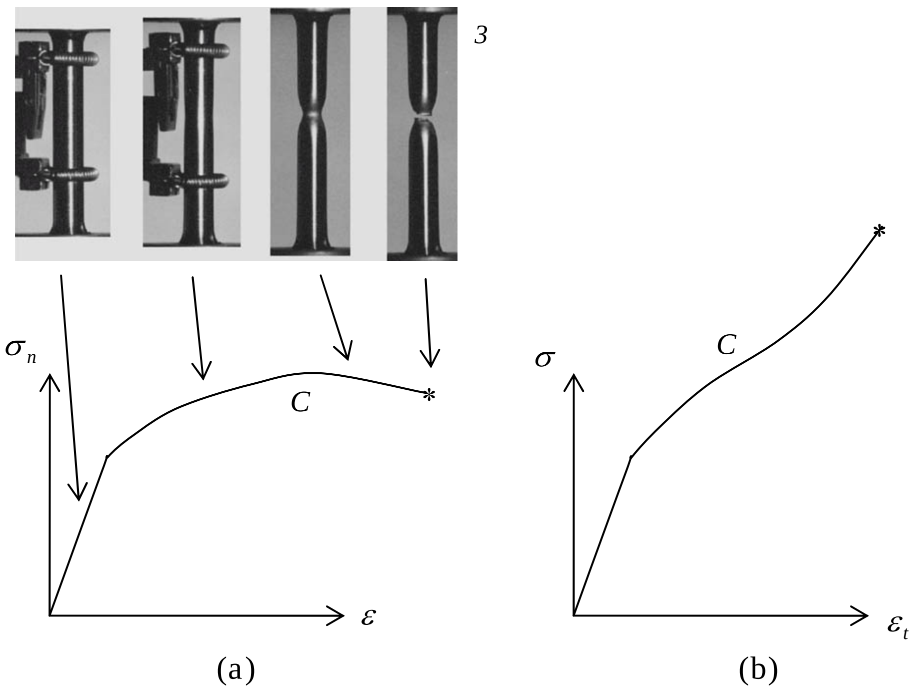

Figure 5.2.3: typical stress-strain curve for an engineering material; (a) engineering stress and strain, (b) true stress and strain

The stress-strain curves for mild steel and aluminium are shown in Fig. 5.2.4. For mild steel, the stress at first increases after reaching the elastic limit, but then decreases. The curve contains a distinct **yield point**; this is where a large increase in strain begins to occur with little increase in required stress[3], i.e. little hardening. There is no distinct yield point for aluminium (or, in fact, for most materials), Fig. 5.2.4b. In this case, it is useful to define a **yield strength** (or **offset yield point**). This is the maximum stress that can be applied without exceeding a specified value of permanent strain. This offset strain is usually taken to be 0.1 or 0.2% and the yield strength is found by following a line parallel to the linear portion until it intersects the stress-strain curve.

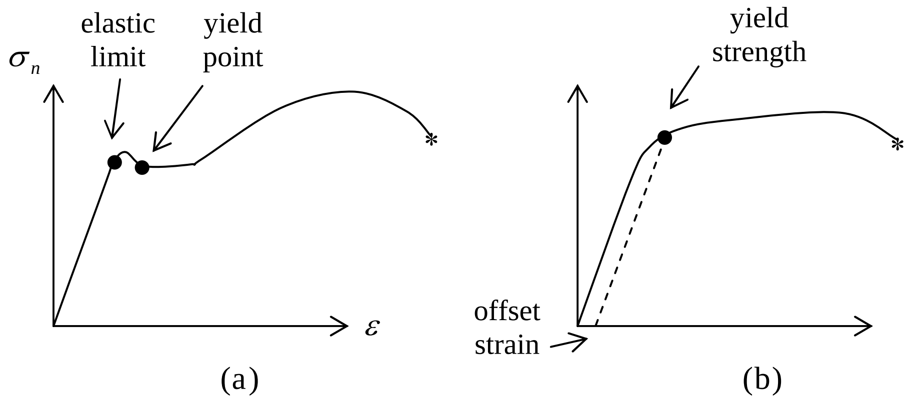

Figure 5.2.4: typical stress-strain curves for (a) mild steel, (b) aluminium

[3] this is also called the **lower yield point**; the **upper yield point** is then the higher stress value just above the elastic limit

The Young's Modulus

The slope of the stress-strain curve over the linear region, before the elastic limit is reached, is the **Young's Modulus** *E*:

$$E = \frac{\sigma}{\varepsilon} \tag{5.2.5}$$

The Young's Modulus has units of stress and is a measure of how "stiff" a material is.

Eqn. 5.2.5 is a constitutive relation (it is of the general form of Eqn. 5.1.1-2); it is the **one-dimensional linear elastic** constitutive relation.

Use of the Tension Test Data

What is the data from the tension test used for? First of all, it is of direct use in many structural applications. Many structures, such as bridges, buildings and the human skeleton, are composed in part of relatively long and slender components. In service, these components undergo tension and/or compression, very much like the test specimen in the tension test. The tension test data (the Young's Modulus, the Yield Strength and the UTS) then gives direct information on the amount of stress that these components can safely handle, before undergoing dangerous straining or all-out failure.

More importantly, the tension test data (and similar test data – see below) can be used to predict what will happen when a component of complex three-dimensional shape is loaded in a complex way, nothing like as in the simple tension test. This can be put another way: one must be able to predict the world around us without having to resort to complex, expensive, time-consuming materials testing – one should be able to use the test data from the tension test (and similar simple tests) to achieve this. How this is actually done is a major theme of mechanics modelling.

Test data for a number of metals are listed in Table 5.2.1 below. Note that although some materials can have similar stiffnesses, for example Nickel and Steel, their relative strengths can be very different.

	Young's Modulus *E* (GPa)	0.2% Yield Strength (MPa)	Ultimate Tensile Strength (MPa)
Ni	200	70	400
Mild steel	203	220	430
Steel (AISI 1144)	210	540	840
Cu	120	60	400
Al	70	40	200
Al Alloy (2014-T651)	73	415	485

Table 5.2.1: Tensile test data for some metals (at room temperature)

Data as listed above should be treated with caution – it should be used only as a rough guide to the actual material under study; the data can vary wildly depending on the purity and precise nature of the material. For example, the tensile strength of glass as found in a typical glass window is about 50MPa. For fine glass fibres as used in fibre-reinforced plastics and composite materials, the tensile strength can be 4000MPa. In fact, glass is a good reminder as to why the tensile values differ from material to material – it is due to the difference in microstructure. The glass window has many very fine flaws and cracks in it, invisible to the naked eye, and so this glass is not very strong; very fine slivers of glass have no such flaws and are extremely strong – hence their use in engineering applications.

The Poisson's Ratio

Another useful material parameter is the **Poisson's ratio** ν.[4] As the material stretches in the tension test, it gets thinner; the Poisson's ratio is a measure of the ease with which it thins:

$$\nu = -\frac{\Delta w / w_0}{\Delta l / l_0} = -\frac{\varepsilon_w}{\varepsilon} \tag{5.2.6}$$

Here, $\Delta w = w - w_0$, w_0 are the change in thickness and original thickness of the specimen, Fig. 5.2.5; $\Delta l = l - l_0$, l_0 are the change in length and original length of the specimen; $\varepsilon_w = (w - w_0)/w_0$ is the strain in the thickness direction. A negative sign is included because Δw is negative, making the Poisson's ratio a positive number. (It is implicitly assumed here that the material is getting thinner by the same amount in all directions; see below in the context of anisotropy for when this is not the case.)

Most engineering materials have a Poisson's Ratio of about 0.3. Values for a range of materials are listed in Table 5.2.2 further below.

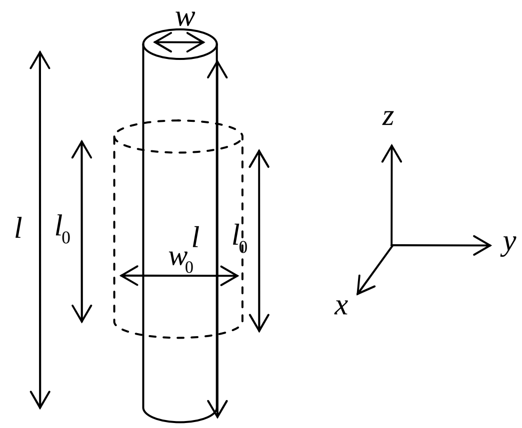

Figure 5.2.5: Change in dimensions of a test specimen

Recall from Section 4.3 that the volumetric strain is given by the sum of the normal strains. There is no harm in re-calculating this for the tensile test specimen of Fig. 5.2.5. One has $\Delta V / V = w^2 l / w_0^2 l_0 - 1$, so that, assuming the strains are small so that the terms $\varepsilon\varepsilon_w$, ε_w^2 and $\varepsilon\varepsilon_w^2$ can be neglected, $\Delta V / V = \varepsilon + 2\varepsilon_w$ (this is the sum of the normal

[4] this is the Greek letter *nu*, not the letter "v"

strains, $\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$, Fig. 5.2.5). Using the definition of the Poisson's ratio, Eqn. 5.2.6, one has

$$\frac{\Delta V}{V} = \varepsilon(1-2\nu) \tag{5.2.7}$$

A material which undergoes little volume change thus has a Poisson's Ratio close to 0.5; rubber and other soft tissues, for example biological materials, have Poisson's Ratios very close to 0.5. A material which undergoes zero volume change ($\nu = 0.5$) is called **incompressible** (see more on incompressibility in Section 5.2.4 below). At the other extreme, materials such as cork can have Poisson's Ratios close to zero. The reason for this can be seen from the microstructure of cork shown in Fig. 5.2.6; when tested in compression, the hexagonal honeycomb structure simply folds down, with no necessary lateral expansion.

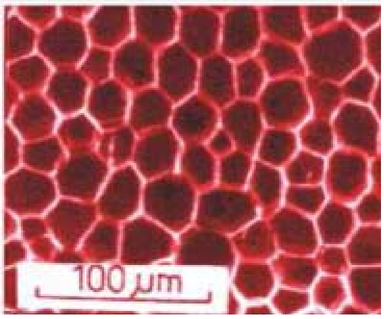

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Figure 5.2.6: Microstructure of Cork

Auxetic materials are materials which have a negative Poisson's Ratio; when they are stretched, they get thicker. Examples can be found amongst polymers, foams, rocks and biological materials. These materials obviously have a very peculiar microstructure. A typical example is the network microstructure shown in Fig. 5.2.7.

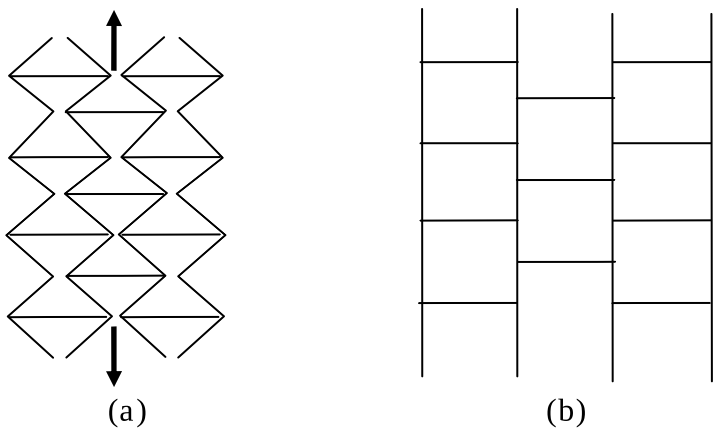


Figure 5.2.7: Auxetic material (a) before loading, (b) after loading

Ductile and Brittle Materials

The engineering materials can be grouped into two broad classes: the ductile materials and the brittle materials. The ductile materials undergo large permanent deformations, stretching and necking before failing[5]. The term ductile **rupture** is usually reserved for materials which fail in this way. The separate pieces of the specimen pull away from each other gradually, leaving rough surfaces. A simple measure of ductility is the engineering strain at failure. The brittle materials are generally more stiff and strong, but fail without undergoing much permanent deformation – the tension specimen undergoes a sudden clean break – a **fracture**. The UTS in the case of a brittle material is the same as the failure/fracture stress. Ceramics and glasses are extremely brittle – they fractures suddenly without undergoing any permanent deformation. The difference is illustrated schematically in Fig. 5.2.8 below.

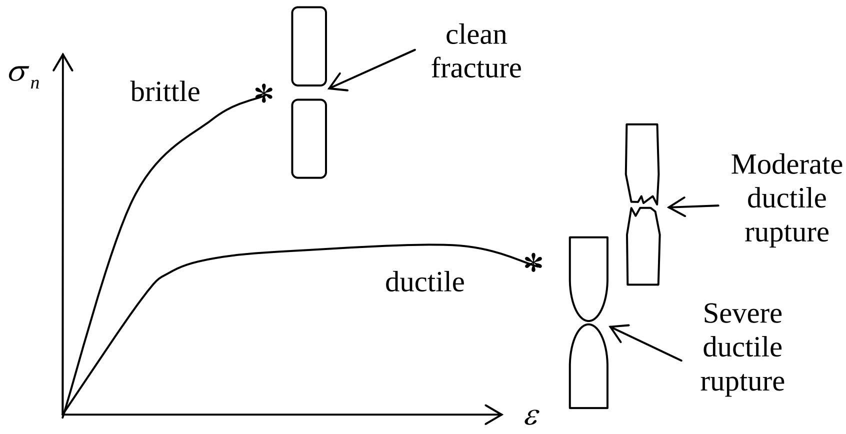


Figure 5.2.8: the difference between ductile and brittle materials

Ductility will depend on temperature – a very cold metal will tend to shatter suddenly, whereas it will stretch more easily when hot.

Soft Materials

Tension test data for (the traditionally) non-engineering materials can be very different to that given above. For example, the typical response of a "soft" material, such as rubber, is shown in Fig. 5.2.9. For many soft materials, the elastic limit (or yield strength) can be very high on the stress-strain curve, close to failure. Most of the curve is elastic, meaning that when one unloads the material, the unloading curve traces over the loading curve back down to zero stress and zero strain: the material does not undergo any permanent deformation[6]. Note that the stress-strain curve is non-linear (curved), unlike the straight line elastic portion for a typical metal, Fig. 5.2.2-4, so these materials do not have a single Young's Modulus through which their response can be described.

[5] the term ductile is used for a specimen in tension; the analogous term for compression is **malleability** – a malleable material is easily "squashed"
[6] here, as elsewhere, these statements should not be taken literally; a real rubber will undergo *some* permanent deformation, only it will often be so small that it can be discounted, and an unload curve will never "exactly" trace over a loading curve

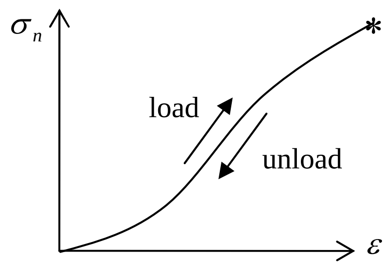
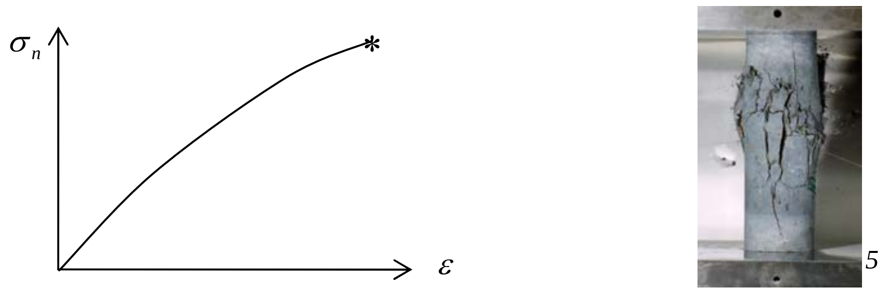


Figure 5.2.9: typical load/unload curves for rubber

5.2.2 Compression Tests

Many materials are used, or designed for use, in compression only, for example soils and concrete. These materials are tested in compression. A common testing method for concrete is to place a cylindrical specimen between two parallel plates and bring the plates together. The typical response of concrete is shown in Fig. 5.2.10a; at failure, the concrete crushes catastrophically, as in the specimen shown in Fig. 5.2.10b. Nominal stresses in the region 20-70MPa are typical and a good concrete would strain to much less than 1% at failure.



(a) (b)

Figure 5.2.10: typical compressive response of concrete; (a) stress-strain curve, (b) specimen at failure

For many materials, e.g. metals, a compression test will lead to similar results as the tensile stress. The yield strength in compression will be approximately the same as (the negative of) the yield strength in tension. If one plots the true stress versus true strain curve for both tension and compression (absolute values for the compression), and the two curves more or less coincide, this would indicate that the behaviour of the material under compression is broadly similar to that under tension. However, if one were to use the nominal stress and strain, then the two curves would not coincide even if the real tensile/compressive behaviour was similar (although they would of course in the small-strain linear region); this is due to the definition of the engineering strain/stress.

5.2.3 Shear Tests

In the **shear test**, the material is subjected to a shear strain $\gamma \equiv 2\varepsilon_{xy}$ by applying a shear stress[7] $\tau \equiv \sigma_{xy}$, Fig. 5.2.11a. The resulting shear stress-strain curve will be similar to the tensile stress-strain curve, Fig. 5.2.11b. The shear stress at failure, the **shear strength**, can be greater or smaller than the UTS. The shear yield strength, on the other hand, is usually in the region of 0.5-0.75 times the tensile yield strength. In the linear small-strain region, the shear stress will be proportional to the shear strain; the constant of proportionality is the **shear modulus** G:

$$G = \frac{\tau}{\gamma} \tag{5.2.8}$$

For many of the engineering materials, $G \approx 0.4E$.

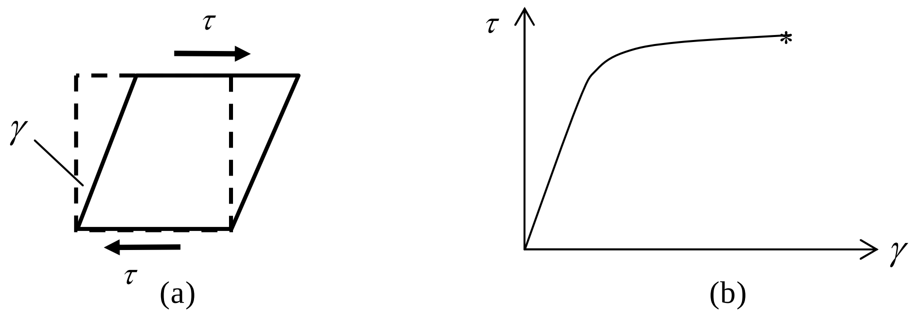


Figure 5.2.11: the shear test; (a) specimen subjected to shear stress and shear strain (dotted = undeformed), (b) shear stress-strain curve

5.2.4 Compressibility

In the **confined compression test**, a sample is placed in a container and a piston is used to compress it at some pressure p, Fig. 5.2.12a. This test can be used to determine how compressible a material is. When a material is compressed by equal pressures on all sides, the ratio of applied pressure p to (unit) volume change, i.e. volumetric strain $\Delta V / V$, is called the **Bulk Modulus** K, Fig. 5.2.12b (this is not quite the situation in Fig. 5.2.12a – the reaction pressures on the side walls will only be about half the applied surface pressure p; see Section 6.2):

$$K = -\frac{p}{\Delta V / V} \tag{5.2.9}$$

The negative sign is included since a positive pressure implies a negative volumetric strain, so that the Bulk Modulus is a positive value.

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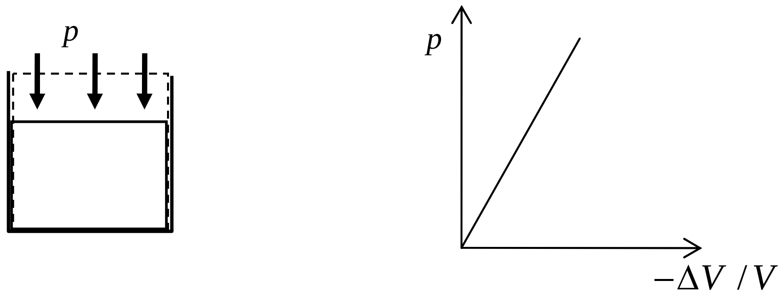
[7] there are many ways that this can be done, for example by pushing blocks of the material over each other, or using more sophisticated methods such as twisting thin tubes of the material (see Section 7.2)



(a) (b)

Figure 5.2.12: the confined compression test; (a) specimen subjected to confined compression, (b) pressure plotted against volume change

A material which can be easily compressed has a low Bulk Modulus. As mentioned earlier, a material which cannot be compressed at all is called incompressible ($K \to \infty$).

No real material is incompressible, but some can be regarded as incompressible so as to make the mechanics modelling easier. For example, the Shear Modulus of rubber is very much smaller than its Bulk Modulus, Table 5.2.2. Essentially, this means that the shape of rubber can be easily changed as compared to its volume. Thus, in applications where a rubber component is being deformed or subjected to arbitrary stressing, it is perfectly reasonable to simply assume that rubber is incompressible. The same applies, only more so, to water; the Shear Modulus is effectively zero and there is no resistance to change in shape (which will be observed on pouring a glass of water on to the ground); it is thus regarded almost always as completely incompressible. On the other hand, even though the Bulk Modulus of the metals and other engineering materials is very much *larger* than that of water or rubber, they are still regarded as compressible in applications – the extremely small changes in volume are significant.

	Young's Modulus *E* (GPa)	Shear Modulus G (GPa)	Bulk Modulus *K* (GPa)	Poissons Ratio
Ni	200	76	180	0.31
Mild steel	203	78	138	0.30
Steel (AISI 1144)	210	80	140	0.31
Cu	120	46	142	0.34
Al	70	26	76	0.35
Rubber	14.9×10^{-4}	5×10^{-4}	1	0.49
Water	$\approx 10^{-14}$	$\approx 10^{-14}$	2.2	

Table 5.2.2: Moduli and Poisson's Ratios for a number of materials

5.2.5 Cyclic Tests

Many materials are subjected to complex loading regimes when in service, not simply a one-off stretching, shearing or compression. A classic example are the wings of an aircraft which are continually loaded in tension, then compression, then tension and so on, as in Fig. 5.2.13. Another example would be the stresses experienced by cardiac tissue in a pumping heart. Anything moving back and forward is likely to be subjected to this tension/compression-type cyclic loading.

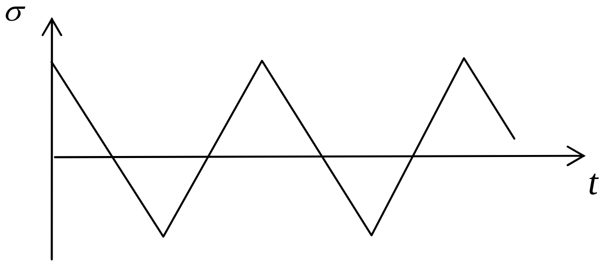
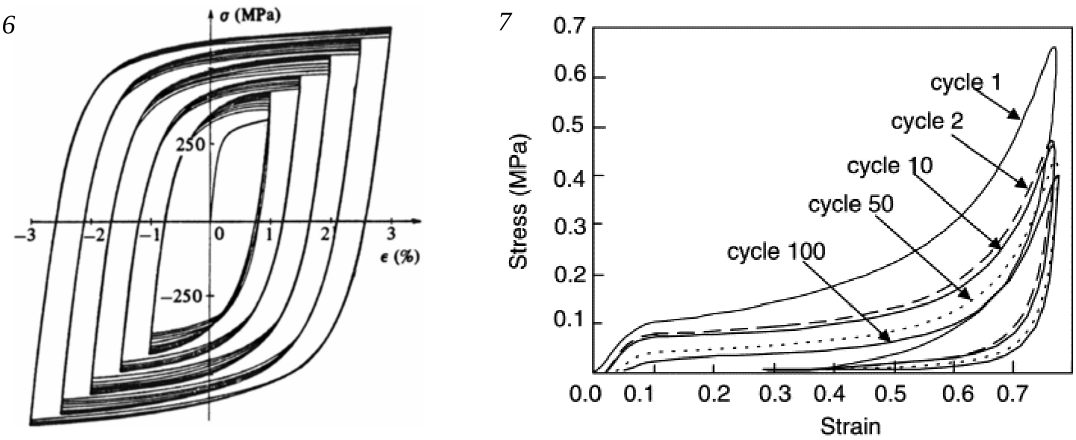


Figure 5.2.13: cyclic loading; alternating between tension (positive stress) and compression (negative stress) over time *t*

Cyclic tests can be carried out to determine the response of materials to such loading cycles. An example is shown in Fig. 5.2.14a, the stress-strain response of a Stainless Steel. The Steel is first cycled between two strain values (one positive, one negative, differing only in sign) a number of times. The stress is seen to increase on each successive cycle. The strain is then increased for a number of further cycles, and so on.

One does not have to move from tension to compression; many materials cycle in only tension or compression. For example, the response to cyclic (compressive) loading of polyurethane foam is shown in Fig. 5.2.14b (note how the loading curve is similar to that in 5.2.9).

[6]



(a) (b)

Figure 5.2.14: cyclic loading; (a) cyclic straining of a Stainless Steel, (b) cyclic loading (in compression) of a polyurethane foam

5.2.6 Other Tests

There are other important tests, for example the Vickers and Brinell **hardness tests**, and the **three-point bending test** (the bending test is discussed in section 7.4.9, in the context of beam theory). Another two very important tests, the **creep test** and the **stress relaxation test**, will be discussed in Chapter 10.

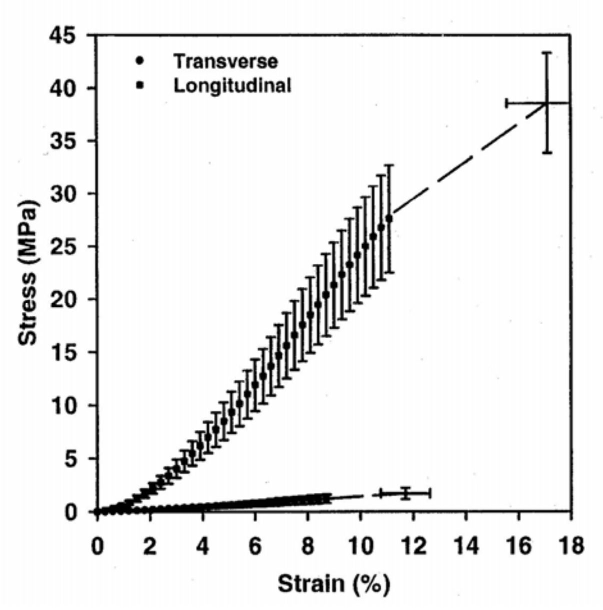
5.2.7 Isotropy and Anisotropy

Many materials have a strong direction-dependence. The classic example is wood, which has a clear structure –along the grain, along which fine lines can be seen, and against the grain, Fig. 5.2.15. The wood is stiffer and stronger along the grain than against the grain. A material which has this direction-dependence of mechanical (and physical) properties is called **anisotropic**.


8

Figure 5.2.15: Wood

Fig. 5.2.16 shows stress-strain curves for human ligament tissue; in one test, the ligament is stretched along its length (the **longitudinal** direction), in the second, across the width of the ligament (the **transverse** direction). It can be seen that the stiffness is much higher in the longitudinal direction. Another example is bone – it is much stiffer along the length of the bone than across the width of the bone. In fact, many biological materials are strongly anisotropic.



9

Figure 5.2.16: Anisotropic response of human ligament

A material whose properties are the same in all directions is called **isotropic**. In particular, the relationship between stress and strain *at any single location* in a material is the same in all directions. This implies that if a specimen is cut from an isotropic material and subjected to a load, it would not matter in which orientation the specimen is cut, the resulting deformation would be the same – as illustrated in Fig. 5.2.17. Most metals and ceramics can be considered to be isotropic (see Section 5.4).

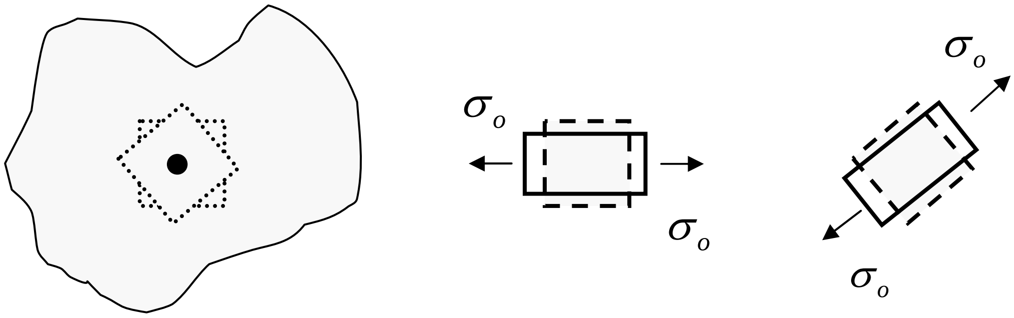


Figure 5.2.17: Illustration of Isotropy; the relationship between stress and strain is the same no matter in what "direction" the test specimen is cut from the material

Anisotropy will be examined in more detail in §6.3. It will be shown there, for example, that an anisotropic material can have a Poisson's ratio greater than 0.5.[8]

5.2.8 Homogeneous Materials

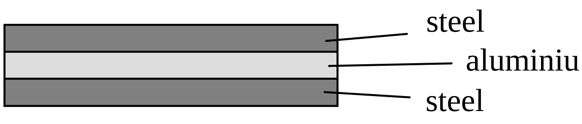
The term homogeneous means that the mechanical properties are the same at each point throughout the material. In other words, the relationship between stress and strain is the same for all material particles. Most materials can be assumed to be homogeneous.

In engineering applications, it is sometimes beneficial to design materials/components which are specifically not homogeneous, i.e. **inhomogeneous**. Such materials whose properties vary gradually throughout are called **Functionally Graded Materials**, and have been gaining popularity since the 1980s-90s in advanced technologies.

Note that a material can be homogeneous and not isotropic, and *vice versa* – homogeneous refers to different locations whereas isotropy refers to the same location.

5.2.9 Problems

1. Steel and aluminium can be considered to be isotropic and homogeneous materials. Is the composite sandwich-structure shown here isotropic and/or homogeneous? Everywhere in the sandwich?



[8] cork was mentioned earlier and it was pointed out that it has a near-zero Poisson's Ratio; actually, cork is quite anisotropic and the Poisson's Ratio in other "directions" will be different (close to 1.0)

Images used:

1. http://site.metacos.com/main/3108/index.asp?pageid=84386&t=&AlbumID=0&page=2
2. http://travisarp.wordpress.com/2012/01/24/how-do-they-do-that-tenderness/
3. http://www.ara.com/Projects/SVO/popups/weld_geometry.html
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5.3 Material Models

The response of real materials to various loading conditions was discussed in the previous section. Now comes the task of creating mathematical models which can predict this response. To this end, it is helpful to categorise the material responses into ideal models. There are four broad **material models** which are used for this purpose: (1) the **elastic model**, (2) the **viscoelastic model**, (3) the **plastic model**, and (4) the **viscoplastic model**. These models will be discussed briefly in what follows, and in more depth throughout the rest of this book.

5.3.1 The Elastic Model

An ideal elastic material has the following characteristics:

(i) the unloading stress-strain path is the same as the loading path
(ii) there is no dependence on the rate of loading or straining
(iii) it does not undergo permanent deformation; it returns to its precise original shape when the loads are removed

Typical stress-strain curves for an ideal elastic model subjected to a tension (or (compression) test are shown in Fig. 5.3.1. The response of a **linear elastic material**, where the stress is *proportional* to the strain, is shown in Fig. 5.3.1a and that for a **non-linear elastic material** is shown in Fig. 5.3.1b.

From the discussion in the previous section, the linear elastic model will well represent the engineering materials up to their elastic limit (see, for example, Figs. 5.2.2-4). It will also represent the complete stress-strain response up to the point of fracture of many very brittle materials. The model can also be used to represent the response of almost any material, provided the stresses are sufficiently small.

The non-linear elastic model is useful for predicting the response of soft materials like rubber and biological soft tissue (see, for example Fig. 5.2.9).

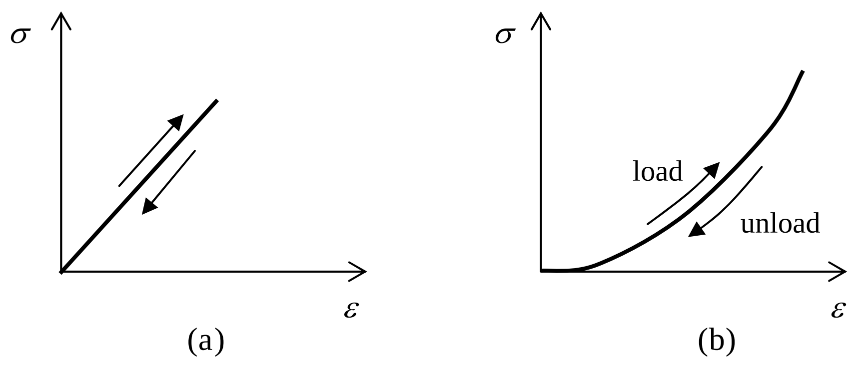

Figure 5.3.1: The Elastic Model; (a) linear elastic, (b) non-linear elastic

It goes without saying that there is no such thing as a purely elastic material. All materials will undergo at least some permanent deformations, even at low loads; no material's response will be exactly the same when stretched at different speeds, and so on.

However, if these occurrences and differences are small enough to be neglected, the ideal elastic model will be useful.

Note also that a prediction of a material's response may be made with accuracy using the elastic model in some circumstances, but not in others. An example would be metal; the elastic model might be able to predict the response right up to high stress levels when the metal is cold, but not so well when the temperature is high, when inelastic effects may not be so easily disregarded (see below).

5.3.2 Viscoelasticity

When solid materials have some "fluid-like" characteristics, they are said to be viscoelastic. A fluid is something which flows easily when subjected to loading – it cannot keep to any particular shape. If a fluid is one (the "viscous") extreme and the elastic solid is at the other extreme, then the viscoelastic material is somewhere in between.

The typical response of a viscoelastic material is sketched in Fig. 5.3.2. The following will be noted:

(i) the loading and unloading curves do not coincide, Fig. 5.3.2a, but form a **hysteresis loop**
(ii) there is a dependence on the rate of straining $d\varepsilon / dt$, Fig. 5.3.2b; the faster the stretching, the larger the stress required
(iii) there may or may not be some permanent deformation upon complete unloading, Fig. 5.3.2a

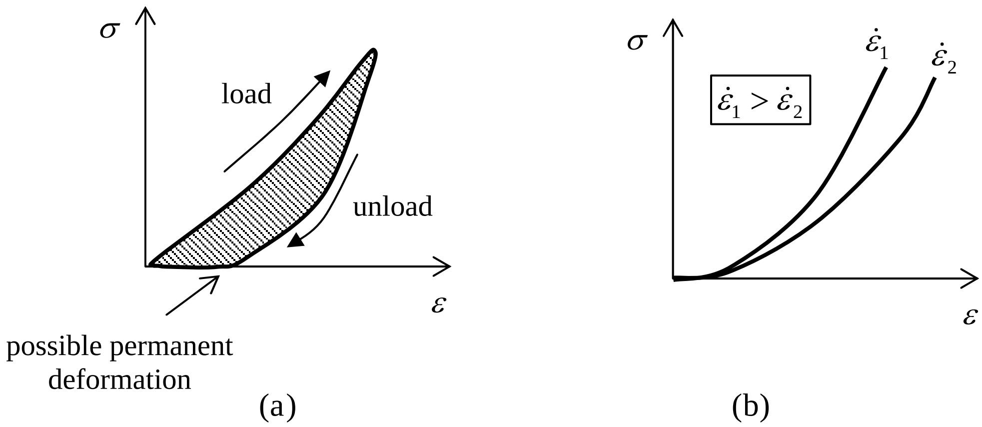

Figure 5.3.2: Response of a Viscoelastic material in the Tension test; (a) loading and unloading with possible permanent deformation (non-zero strain at zero stress), (b) different rates of stretching

The effect of *rate* of stretching shows that the viscoelastic material *depends on time*. This contrasts with the elastic material; it makes no difference whether an elastic material is loaded to some given stress level for one second or one day, or quickly or slowly, the resulting strain will be the same. This rate effect can be seen when you push your hand through water – it is easier to do so when you push slowly than when you push fast.

Depending on how "fluid-like" or "solid-like" a material is, it can be considered to be a **viscoelastic fluid**, for example blood or toothpaste, or a **viscoelastic solid**, for example Silly Putty™ or foam. That said, the model for both and the theory behind each will be similar.

Viscoelastic materials will be discussed in detail in Chapter 10.

5.3.3 Plasticity

Plasticity has the following characteristics:
(i) The loading is elastic up to some threshold limit, beyond which permanent deformations occur
(ii) The permanent deformation, i.e. the **plasticity**, is time independent

This plasticity can be seen in Figs. 5.2.2-4. The threshold limit – the elastic limit – can be quite high but it can also be extremely small, so small that significant permanent deformations occur at almost any level of loading. The plasticity model is particularly useful in describing the permanent deformations which occur in metals, soils and other engineering materials. It will be discussed in further detail in Chapter 11.

5.3.4 Viscoplasticity

Finally, the viscoplastic model is a combination of the viscoelastic and plastic models. In this model, the plasticity is rate-dependent. One of the main applications of the model is in the study of metals at high temperatures, but it is used also in the modeling of a huge range of materials and other applications, for example asphalt, concrete, clay, paper pulp, biological cells growth, etc. This model will be discussed in Chapter 12.

5.4 Continuum Models and Micromechanics

The models mentioned in the previous section are **continuum models**. What this means is explained in what follows.

5.4.1 Stress and Scale

In the definition of the traction vector, §3.3.1, it was assumed that the ratio of force over area would reach some definite limit as the area ΔS of the surface upon which the force ΔF acts was shrunk to zero. This issue can be explored further by considering Fig. 5.4.1. Assume first that the plane upon which the force acts is fairly large; it is then shrunk and the ratio F/S tracked. A schematic of this ratio is shown in Fig. 5.4.2. At first (to the right of Fig. 5.4.2) the ratio F/S undergoes change, assuming the stress to vary within the material, as it invariably will if the material is loaded in some complex way. Eventually the plane will be so small that the ratio changes very little, perhaps with some small variability ε. If the plane is allowed to get too small, however, down below some distance h^* say and down towards the atomic level, where one might encounter "intermolecular space", there will be large changes in the ratio and the whole concept of a force acing on a single surface breaks down.

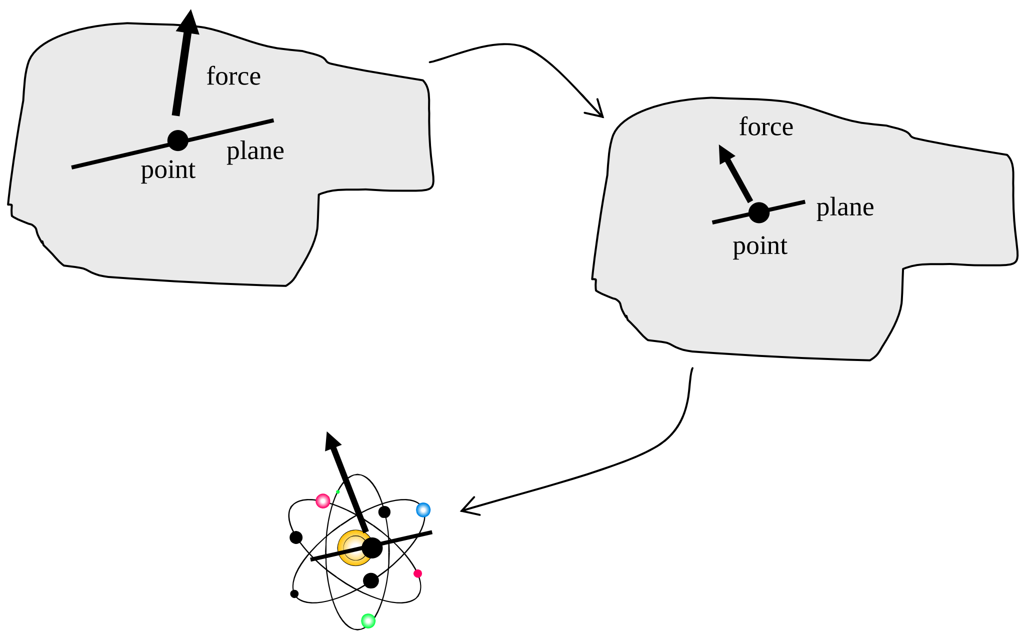

Figure 5.4.1: A force acting on an internal surface; allowing the plane on which the force acts to get progressively smaller

In a continuum model, it is assumed that the ratio F/S follows the dotted path shown in Fig. 5.4.2; a definite limit is reached as the plane shrinks to *zero size*. It should be kept in mind that the traction in a *real* material should be evaluated through

$$\mathbf{t} = \lim_{\Delta S \to (h^*)^2} \frac{\Delta F}{\Delta S} \tag{5.4.1}$$

where h^* is some minimum dimension below which there is no acceptable limit. On the other hand, it is necessary to take the limit to zero in the *mathematical* modelling of materials since that is the basis of calculus[1].

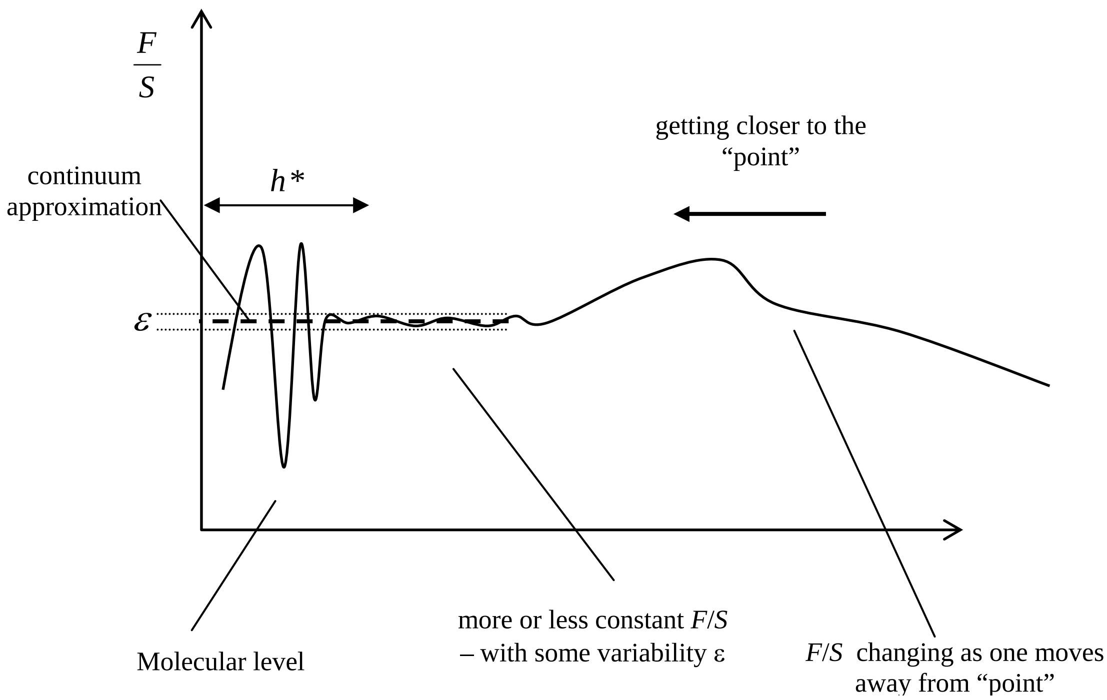

Figure 5.4.2: the change in traction as the plane upon which a force acts is reduced in size

In a continuum model, then, there is a minimum sized element one can consider, say of size $\Delta V = (h^*)^3$. When one talks about the stress on this element, the mass of this element, the density, velocity and acceleration of this element, one means the average of these quantities throughout or over the surface of the element – the discrete atomic structure within the element is ignored and is averaged out, or "smeared" out, into a **continuum element**.

The continuum element is also called a **representative volume element** (RVE), an element of material large enough for the heterogeneities to be replaced by homogenised mean values of their properties. The order of the dimensions of RVE's for some common engineering materials would be approximately (see the metal example which follows)

Metal:	0.1mm
Polymers/composites:	1mm
Wood:	10mm
Concrete:	100mm

One does not have any information about what is happening inside the continuum element – it is like a "black box". The scale of the element (and higher) is called the

[1] calculus is not used too much in this book – it is absolutely necessary and ubiquitous in more refined and advanced mechanics theories

macroscale – continuum mechanics is mechanics on the macroscale. The scale of entities within the element is termed the **microscale** – continuum models cannot give any information about what happens on the microscale.

5.4.2 Example: Metal

Metal, from a distance, appears fairly uniform. With the help of a microscope, however, it will be seen to consist of many individual grains of metal. For example, the metal shown in Fig. 5.4.3 has grains roughly 0.05mm across, and each one has very individual properties (the crystals in each grain are aligned in different directions).

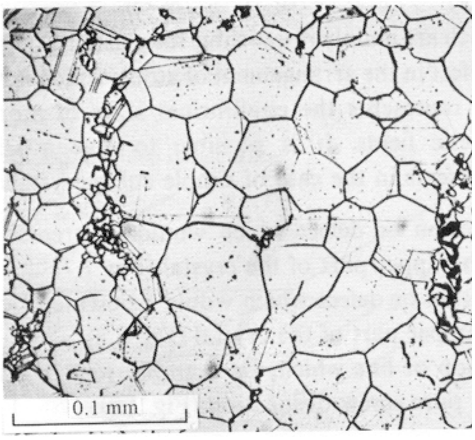

Figure 5.4.3: metal grains

If one is interested in the gross deformation of a moderately sized component of this metal, it would be sufficient to consider deformations that are averaged over volumes which are large compared to individual grains, but small compared to the whole component. A minimum dimension of, say, $h^* = 0.5\text{mm}$ for the metal of Fig. 5.4.3 would seem to suffice, and this would be the macro/micro-scale boundary, with a minimum surface area of dimension $(h^*)^2$ for the definition of stress.

When one measures physical properties of the metal "at a point", for example the density, one need only measure an average quantity over an element of the order, say, $(0.5\text{mm})^3$ or higher. It is not necessary to consider the individual grains of metal – these are inside the "black box". The model will return valuable information about the deformation of the gross material, but it will not be able to furnish any information about movement of individual grains.

It was shown how to evaluate the Young's Modulus and other properties of a metal in Section 5.2.1. The test specimens used for such tests are vastly larger than the continuum elements discussed above. Thus the test data is perfectly adequate to describe the response of the metal, on the macroscale.

What if the response of individual grains to applied loads is required? In that case a model would have to be constructed which accounted for the different mechanical properties of each grain. The metal could no longer be considered to be a uniform material, but a complex one with many individual grains, each with different properties and orientation. The macro/micro boundary could be set at about $h^* = 0.1\mu\text{m}$. There are now two problems which need to be dealt with: (1) experiments such as the tensile test would have to be conducted on specimens much smaller than the grain size in order to provide data for any mathematical model, and (2) the mathematical model will be more complex and difficult to solve.

5.4.3 Micromechanical Models

Consider the schematic of a continuum model shown in Fig. 5.4.4 below. One can determine the material's properties, such as the Young's modulus *E*, through experimentation, and the resulting mathematical continuum model can be used to make predictions about the material's response. With the improved power of computers, especially since the 1990s, it has now become possible to complement continuum models with **micromechanical models**. These models take into account more fine detail of the material's structure (for example of the individual grains of the metal discussed earlier). Usually, one will have a micromechanical model of a small (typical) RVE of material. This then provides information regarding the properties of the RVE to be included in a continuum model (rather than having a micromechanical model of the *complete* material, which is in most cases still not practical). The means by which the properties at the micro scale are averaged (for example into a "smeared out" single *E* value) and passed "up" to the continuum model is through **homogenisation theory**. Such micromechanical models can provide further insight into material behaviour than the simpler continuum model.

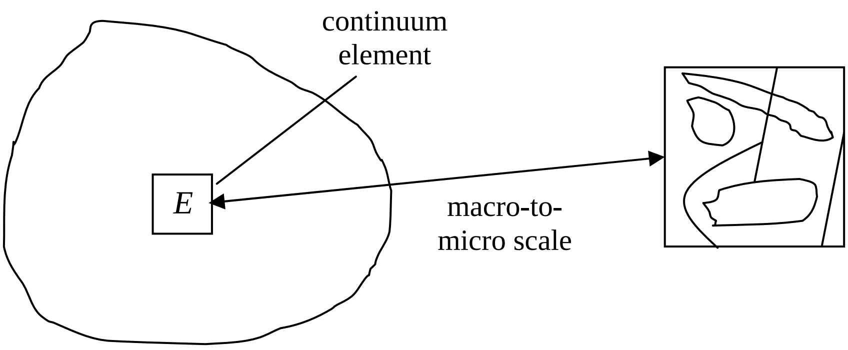

Figure 5.4.4: continuum model and micromechanical model

5.4.4 Problems

1. You want to evaluate the stiffness *E* of a metal for inclusion in a mechanics model. What *minimum* size specimen would you use in your test - $10\mu\text{m}$, 0.1mm, 5mm or 5cm?

2. Individual rice grains are separate solid particles. However, rice flowing down a chute at a food processing plant can be considered to be a fluid, and the flow of rice can be solved using the equations of mechanics. What minimum dimension h^*

should be employed for measurements in this case to ensure the validity of a continuum model of flowing rice?