4 Strain

The concept of strain is introduced in this Chapter. The approximation to the **true strain** of the **engineering strain** is discussed. The practical case of two dimensional **plane strain** is discussed, along with the **strain transformation formulae**, **principal strains**, **principal strain directions** and the **maximum shear strain**.

4.1 Strain

If an object is placed on a table and then the table is moved, each material particle moves in space. The particles undergo a **displacement**. The particles have moved in space as a **rigid body**. The material remains unstressed. On the other hand, when a material is acted upon by a set of forces, it *changes size and/or shape*, it **deforms**. This deformation is described using the concept of **strain**. The study of this movement and deformation, without reference to the forces or anything else which might "cause" it, is called **kinematics**.

4.1.1 One Dimensional Strain

The Engineering Strain

Consider a slender rod, fixed at one end and stretched, as illustrated in Fig. 4.1.1; the original position of the rod is shown dotted.

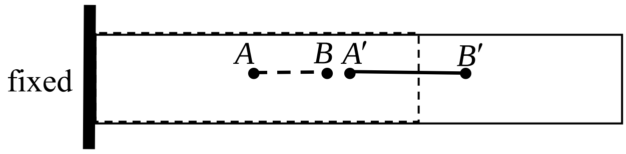


Figure 4.1.1: the strain at a point *A* in a stretched slender rod; *AB* is a line element in the unstretched rod, $A'B'$ is the same line element in the stretched rod

There are a number of different ways in which this stretching/deformation can be described (see later). Here, what is perhaps the simplest measure, the **engineering strain**, will be used. To determine the strain at point *A*, Fig. 4.1.1, consider a small line element *AB* emanating from *A* in the unstretched rod. The points *A* and *B* move to A' and B' when the rod has been stretched. The (engineering) strain ε at *A* is then defined as[1]

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} \tag{4.1.1}$$

The strain at other points in the rod can be evaluated in the same way.

If a line element is stretched to twice its original length, the strain is 1. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.5. The strain is often expressed as a percentage; a 100% strain is a strain of 1, a 200% strain is a strain of 2, etc. Most engineering materials, such as metals and concrete, undergo extremely small strains in practical applications, in the range 10^{-6} to 10^{-2}; rubbery materials can easily undergo large strains of 100%.

[1] this is the strain at point *A*. The strain at *B* is evidently the same – one can consider the line element *AB* to emanate fom point *B* (it does not matter whether the line element emanates out from the point to the "left" or to the "right")

Consider now two adjacent line elements *AE* and *EB* (not necessarily of equal length), which move to $A'E'$ and $E'B'$, Fig. 4.1.2. If the rod is stretching **uniformly**, that is, if all line elements are stretching in the same proportion along the length of the rod, then $|A'E'|/|AE| = |E'B'|/|EB|$, and $\varepsilon^{(A)} = \varepsilon^{(E)}$; the strain is the same at all points along the rod.

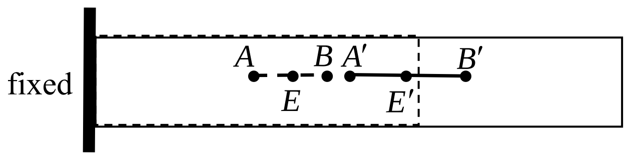


Figure 4.1.2: the strain at a point *A* and the strain at point *E* in a stretched rod

In this case, one could equally choose the line element *AB* or the element *AE* in the calculation of the strain at *A*, since

$$\varepsilon^{(A)} = \frac{|A'B'| - |AB|}{|AB|} = \frac{|A'E'| - |AE|}{|AE|}$$

In other words it does not matter what the length of the line element chosen for the calculation of the strain at *A* is. In fact, if the length of the rod before stretching is L_0 and after stretching it is L, Fig. 4.1.3, the strain everywhere is (this is equivalent to choosing a "line element" extending the full length of the rod)

$$\varepsilon = \frac{L - L_0}{L_0} \tag{4.1.2}$$

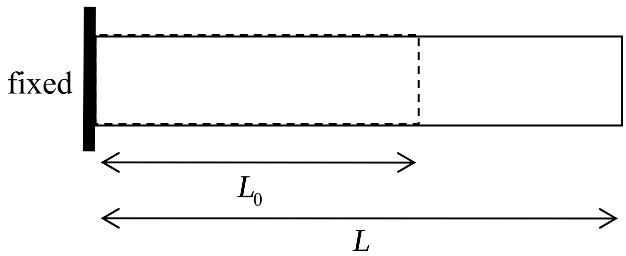


Figure 4.1.3: a stretched slender rod

On the other hand, when the strain is *not* uniform, for example $|A'E'|/|AE| \neq |E'B'|/|EB|$, then the length of the line element does matter. In this case, to be precise, the line element *AB* in the definition of strain in Eqn. 4.1.1 should be "infinitely small"; the smaller the line element, the more accurate will be the evaluation of the strain. The strains considered in this book will be mainly uniform.

Displacement, Strain and Rigid Body Motions

To highlight the difference between displacement and strain, and their relationship, consider again the stretched rod of Fig 4.1.1. Fig 4.1.4 shows the same rod: the two points A and B undergo displacements $u^{(A)} = |AA'|, u^{(B)} = |BB'|$. The strain at A, Eqn 4.1.1, can be re-expressed in terms of these displacements:

$$\varepsilon^{(A)} = \frac{u^{(B)} - u^{(A)}}{|AB|} \tag{4.1.3}$$

In words, the strain is a measure of the *change* in displacement as one moves along the rod.

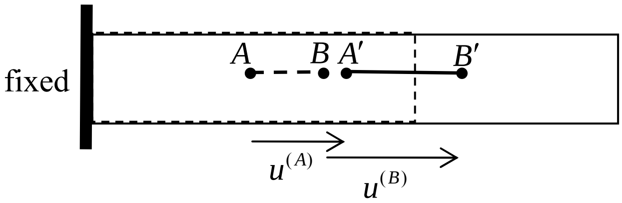


Figure 4.1.4: displacements in a stretched rod

Consider a line element emanating from the left-hand fixed end of the rod. The displacement at the fixed end is zero. However, the strain at the fixed end is *not* zero, since the line element there will change in length. This is a case where the displacement is zero but the strain is not zero.

Consider next the case where the rod is not fixed and simply moves/translates in space, without any stretching, Fig. 4.1.5. This is a case where the displacements are all non-zero (and in this case everywhere the same) but the strain is everywhere zero. This is in fact a feature of a good measure of strain: it should be zero for any rigid body motion; the strain should only measure the deformation.

Figure 4.1.5: a rigid body translation of a rod

Note that if one knows the strain at all points in the rod, one cannot be sure of the rod's exact position in space – again, this is because strain does not include information about possible rigid body motion. To know the precise position of the rod, one must also have some information about the displacements.

The True Strain

As mentioned, there are many ways in which deformation can be measured. Many different strains measures are in use apart from the engineering strain, for example the Green-Lagrange strain and the Euler-Almansi strain: referring again to Fig. 4.1.1, these are

Green-Lagrange $\varepsilon^{(A)} = \dfrac{|A'B'|^2 - |AB|^2}{2|AB|^2}$, Euler-Alamnsi $\varepsilon^{(A)} = \dfrac{|A'B'|^2 - |AB|^2}{2|A'B'|^2}$ (4.1.4)

Many of these strain measures are used in more advanced theories of material behaviour, particularly when the deformations are very large. Apart from the engineering strain, just one other measure will be discussed in any detail here: the **true strain** (or **logarithmic strain**), since it is often used in describing material testing (see Chapter 5).

The true strain may be defined as follows: define a small increment in strain to be the change in length divided by the *current* length: $d\varepsilon_t = dL / L$. As the rod of Fig. 4.1.1 stretches (uniformly), this current length continually changes, and the total strain thus defined is the accumulation of these increments:

$$\varepsilon_t = \int_{L_0}^{L} \frac{dL}{L} = \ln\left(\frac{L}{L_0}\right). \tag{4.1.5}$$

If a line element is stretched to twice its original length, the (true) strain is 0.69. If it is unstretched, the strain is 0. If it is shortened to half its original length, the strain is -0.69. The fact that a stretching and a contraction of the material by the same factor results in strains which differ only in sign is one of the reasons for the usefulness of the true strain measure.

Another reason for its usefulness is the fact that the true strain is additive. For example, if a line element stretches in two steps from lengths L_1 to L_2 to L_3, the total true strain is

$$\varepsilon_t = \ln\left(\frac{L_3}{L_2}\right) + \ln\left(\frac{L_2}{L_1}\right) = \ln\left(\frac{L_3}{L_1}\right),$$

which is the same as if the stretching had occurred in one step. This is not true of the engineering strain.

The true strain and engineering strain are related through (see Eqn. 4.1.2, 4.1.5)

$$\varepsilon_t = \ln(1+\varepsilon) \tag{4.1.6}$$

One important consequence of this relationship is that the smaller the deformation, the less the difference between the two strains. This can be seen in Table 4.1 below, which shows the values of the engineering and true strains for a line element of initial length 1mm, at different stretched lengths. (In fact, using a Taylor series expansion, $\varepsilon_t = \ln(1+\varepsilon) \approx \varepsilon - \frac{1}{2}\varepsilon^2 + \frac{1}{3}\varepsilon^3 - \cdots$, for small ε.) Almost all strain measures in use are similar in this way: they are defined such that they are more or less equal when the deformation is small. Put another way, when the deformations are small, it does not really matter which strain measure is used, since they are all essentially the same – in that case it is often sensible to use the simplest measure.

L_0 (mm)	L (mm)	ε	ε_t
1	2	1	0.693
1	1.5	0.5	0.405
1	1.4	0.4	0.336
1	1.3	0.3	0.262
1	1.2	0.2	0.182
1	1.1	0.1	0.095
1	1.01	0.01	0.00995
1	1.001	0.001	0.000995

Table 4.1: true strain and engineering strain at different stretches

If one defines strain to be "change in length over length", then the true strain would be more "correct" than the engineering strain. On the other hand, if the strain is considered to have a variety of (related) definitions, such as Eqns 4.1.2, 4.1.4-5, then no strain measure is really more "correct" than any other; the usefulness of a strain measure will depend on the application and the problem at hand.

4.1.2 Two Dimensional Strain

The two dimensional case is similar to the one dimensional case, in that material deformation can be described by imagining the material to be a collection of small line elements. As the material is deformed, the line elements stretch, or get shorter, only now they can also rotate in space relative to each other. This movement of line elements is encompassed in the idea of strain: the "strain at a point" is all the stretching, contracting and rotating of *all* line elements emanating from that point, with all the line elements together making up the continuous material, as illustrated in Fig. 4.1.6.

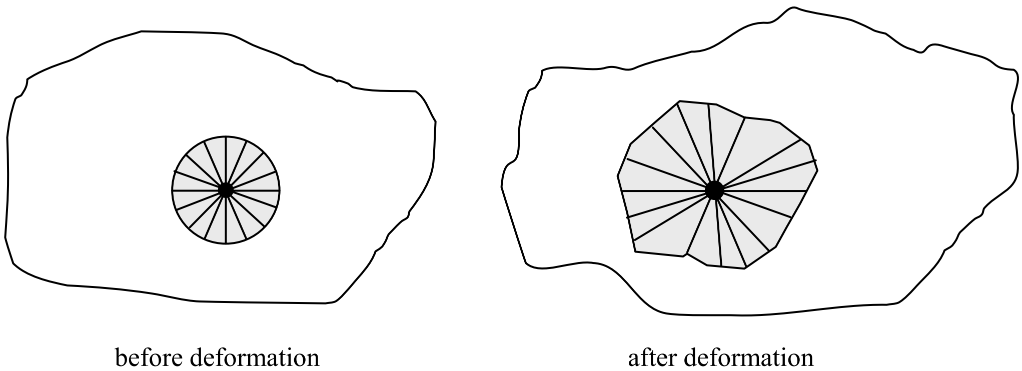


Figure 4.1.6: a deforming material element; original state of line elements and their final position after straining

It turns out that the strain at a point is completely characterised by the movement of *any two mutually perpendicular line-segments*. If it is known how these perpendicular line-segments are stretching, contracting and rotating, it will be possible to determine how any other line element at the point is behaving, by using a **strain transformation rule** (see later). This is analogous to the way the stress at a point is characterised by the stress acting on perpendicular planes through a point, and the stress components on other planes can be obtained using the stress transformation formulae.

So, for the two-dimensional case, consider two perpendicular line-elements emanating from a point. When the material that contains the point is deformed, two things (can) happen:

(1) the line segments will *change length* and
(2) the *angle* between the line-segments *changes*.

The change in length of line-elements is called **normal strain** and the change in angle between initially perpendicular line-segments is called **shear strain**.

As mentioned earlier, a number of different definitions of strain are in use; here, the following, most commonly used, definition will be employed, which will be called the **exact strain**:

Normal strain in direction x: (denoted by ε_{xx})
change in length (per unit length) of a line element originally lying in the x–direction

Normal strain in direction y: (denoted by ε_{yy})
change in length (per unit length) of a line element originally lying in the y–direction

Shear strain: (denoted by ε_{xy})
(half) the change in the original right angle between the two perpendicular line elements

Referring to Fig. 4.1.7, the (exact) strains are

$$\varepsilon_{xx} = \frac{A'B' - AB}{AB}, \quad \varepsilon_{yy} = \frac{A'C' - AC}{AC}, \quad \varepsilon_{xy} = \frac{1}{2}(\theta + \lambda). \qquad (4.1.7)$$

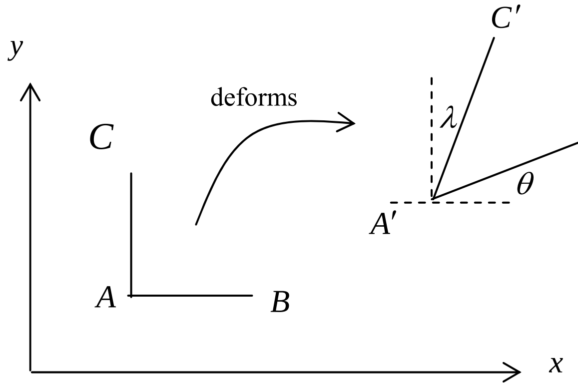


Figure 4.1.7: strain at a point *A*

These 2D strains can be represented in the matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.1.8}$$

As with the stress, the strain matrix is symmetric, with, by definition, $\varepsilon_{xy} = \varepsilon_{yx}$.
Note that the point *A* in Fig. 4.1.7 has also undergone a displacement $\mathbf{u}(A)$. This displacement has two components, u_x and u_y, as shown in Fig. 4.1.8 (and similarly for the points *B* and *C*).

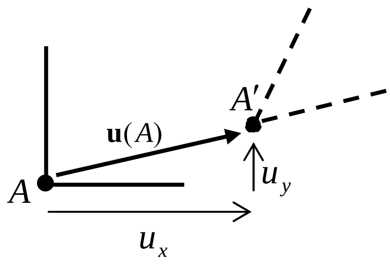


Figure 4.1.8: displacement of a point *A*

The line elements not only change length and the angle between them changes – they can also move in space as rigid-bodies. Thus, for example, the normal and shear strain in the three examples shown in Fig. 4.1.9 are the same, even though the displacements occurring in each case are different – *strain is independent of rigid body motions*.

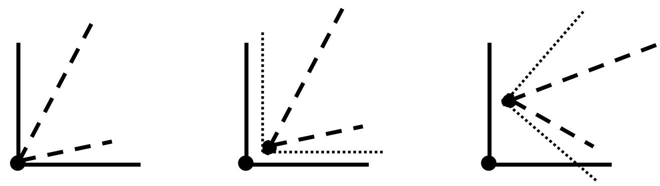


Figure 4.1.9: rigid body motions

The Engineering Strain

Suppose now that the deformation is very small, so that, in Fig. 4.1.10, $A'B' \approx A'B^*$; here $A'B^*$ is the projection of $A'B'$ in the $x-$direction. In that case,

$$\varepsilon_{xx} \approx \frac{A'B^* - AB}{AB}. \tag{4.1.9}$$

Similarly, one can make the approximations

$$\varepsilon_{yy} \approx \frac{A'C^* - AC}{AC}, \quad \varepsilon_{xy} \approx \frac{1}{2}\left(\frac{B^*B'}{AB} + \frac{C^*C'}{AC}\right) \quad , \tag{4.1.10}$$

the expression for shear strain following from the fact that, for a *small angle*, the angle (measured in radians) is approximately equal to the tan of the angle.

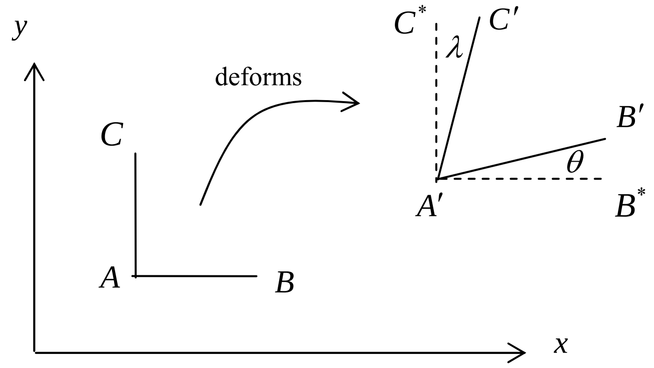


Figure 4.1.10: small deformation

This approximation for the normal strains is called the **engineering strain** or **small strain** or **infinitesimal strain** and is valid when the *deformations are small*. The advantage of the small strain approximation is that the mathematics is simplified greatly.

Example

Two perpendicular lines are etched onto the fuselage of an aircraft. During testing in a wind tunnel, the perpendicular lines deform as in Fig. 4.1.10. The coordinates of the line end-points (referring to Fig. 4.1.10) are:

$$
\begin{array}{ll}
C:(0.0000,1.0000) & C':(0.0025,1.0030) \\
A:(0.0000,0.0000) & A':(0.0000,0.0000) \\
B:(1.0000,0.0000) & B':(1.0045,0.0020)
\end{array}
$$

The exact strains are, from Eqn. 4.1.9, (to 8 decimal places)

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\sqrt{\left|A'B^*\right|^2+\left|B^*B'\right|^2}}{|AB|}-1=0.00450199 \\
\varepsilon_{yy} &= \frac{\sqrt{\left|A'C^*\right|^2+\left|C^*C'\right|^2}}{|AC|}-1=0.00300312 \\
\varepsilon_{xy} &= \frac{1}{2}\left(\arctan\left(\frac{\left|B^*B'\right|}{\left|A'B^*\right|}\right)+\arctan\left(\frac{\left|C^*C'\right|}{\left|A'C^*\right|}\right)\right)=0.00224178
\end{aligned}
$$

The engineering strains are, from Eqns. 4.1.10-11,

$$
\varepsilon_{xx}=\frac{\left|A'B^*\right|}{|AB|}-1=0.0045, \quad \varepsilon_{yy}=\frac{\left|A'C^*\right|}{|AC|}-1=0.003, \quad \varepsilon_{xy}=\frac{1}{2}\left(\frac{\left|B^*B'\right|}{|AB|}+\frac{\left|C^*C'\right|}{|AC|}\right)=0.00225
$$

As can be seen, for the small deformations which occurred, the errors in making the small-strain approximation are extremely small, less than 0.11% for all three strains.

■

Small strain is useful in characterising the small deformations that take place in, for example, (1) engineering materials such as concrete, metals, stiff plastics and so on, (2) linear viscoelastic materials such as many polymeric materials (see Chapter 10), (3) some porous media such as soils and clays at moderate loads, (4) almost any material if the loading is not too high.

Small strain is inadequate for describing large deformations that occur, for example, in many rubbery materials, soft tissues, engineering materials at large loads, etc. In these cases the more precise definition 4.1.7 (or a variant of it) is required. That said, the engineering strain and the concepts associated with it are an excellent introduction to the more involved large deformation strain measures.

In one dimension, there is no distinction between the exact strain and the engineering strain – they are the same. Differences arise between the two in the two-dimensional case when the material shears (as in the example above), or rotates as a rigid body (as will be discussed further below).

Engineering Shear Strain and Tensorial Shear Strain

The definition of shear strain introduced above is the **tensorial shear strain** ε_{xy}. The **engineering shear strain**[2] γ_{xy} is defined as twice this angle, i.e. as $\theta + \lambda$, and is often used in Strength of Materials and elementary Solid Mechanics analyses.

4.1.3 Sign Convention for Strain

A positive normal strain means that the line element is lengthening. A negative normal strain means the line element is shortening.

For shear strain, one has the following convention: when the two perpendicular line elements are both directed in the positive directions (say x and y), or both directed in the negative directions, then a positive shear strain corresponds to a *decrease* in right angle. Conversely, if one line segment is directed in a positive direction whilst the other is directed in a negative direction, then a positive shear strain corresponds to an *increase* in angle. The four possible cases of shear strain are shown in Fig. 4.1.11a (all four shear strains are positive). A box undergoing a positive shear and a negative shear are also shown, in Figs. 4.1.11b,c.

[2] not to be confused with the term *engineering strain*, i.e. *small strain*, used throughout this Chapter

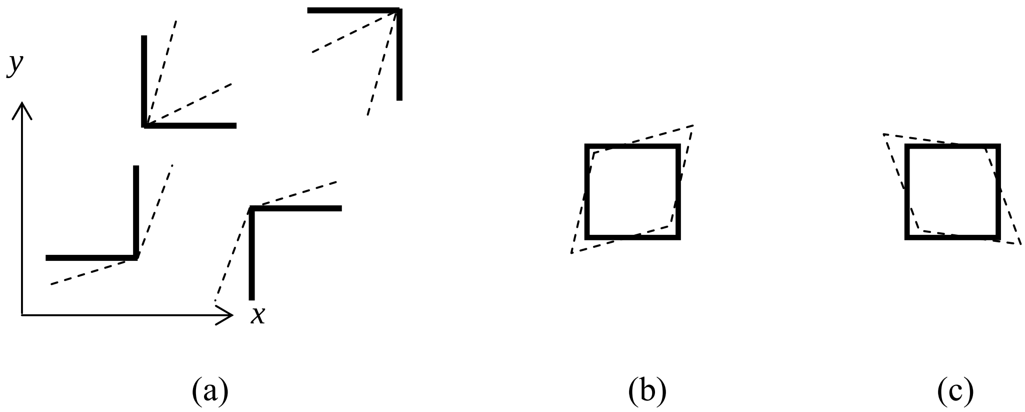


Figure 4.1.11: sign convention for shear strain; (a) line elements undergoing positive shear, (b) a box undergoing positive shear, (c) a box undergoing negative shear

4.1.4 Geometrical Interpretation of the Engineering Strain

Consider a small "box" element and suppose it to be so small that the strain is constant/uniform throughout - one says that the strain is **homogeneous**. This implies that straight lines remain straight after straining and parallel lines remain parallel. A few simple deformations are examined below and these are related to the strains.

A positive normal strain $\varepsilon_{xx} > 0$ is shown in Fig. 4.1.12a. Here the undeformed box element (dashed) has elongated. As mentioned already, knowledge of the strain alone is not enough to determine the position of the strained element, since it is free to move in space as a rigid body. The displacement over some part of the box is usually specified, for example the left hand end has been fixed in Fig. 4.1.12b. A negative normal strain acts in Fig. 4.1.12c.

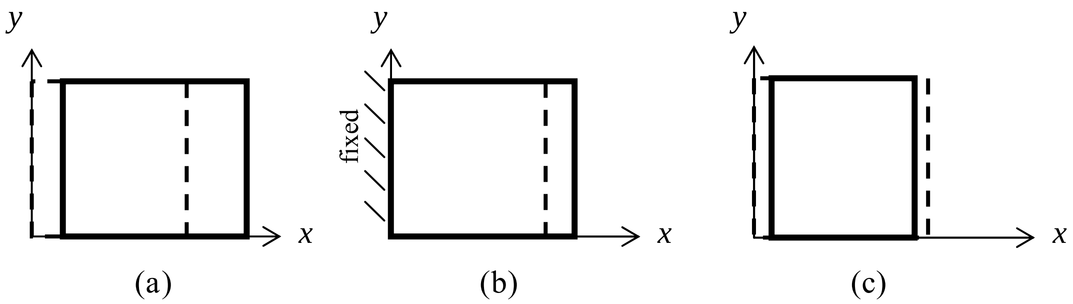


Figure 4.1.12: normal strain; (a) positive normal strain, (b) positive normal strain with the left-hand end fixed in space, (c) negative normal strain

A case known as **simple shear** is shown in Fig. 4.1.13a, and that of **pure shear** is shown in Fig. 4.1.13b. In both illustrations, $\varepsilon_{xy} > 0$. A pure (rigid body) **rotation** is shown in Fig. 4.1.13c (zero strain).

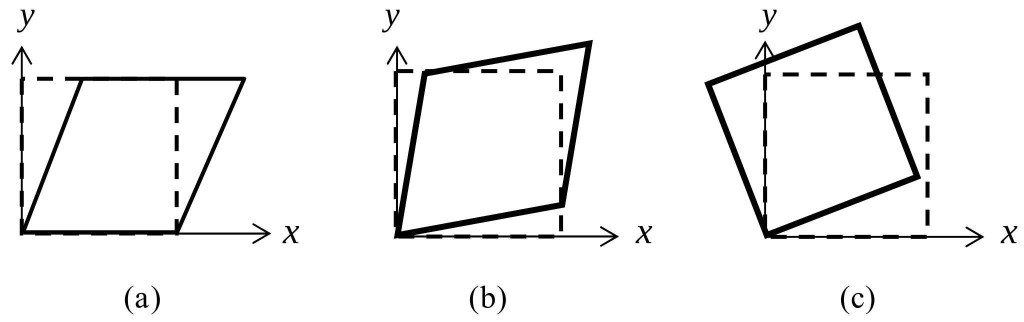


Figure 4.1.13: (a) simple shear, (b) pure shear, (c) pure rotation

Any shear strain can be decomposed into a pure shear and a pure rotation, as illustrated in Fig. 4.1.14.

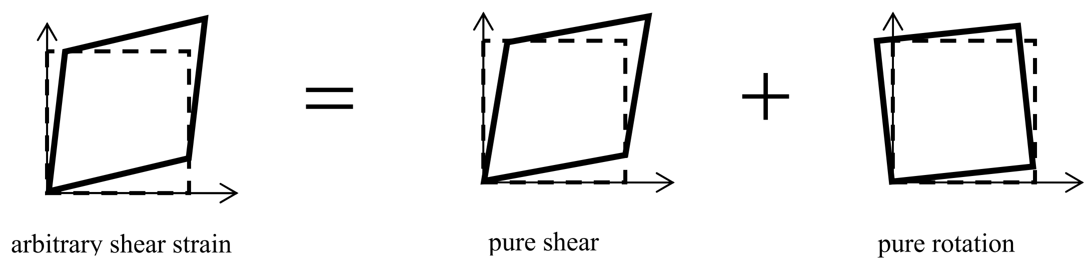


Figure 4.1.14: shear strain decomposed into a pure shear and a pure rotation

4.1.5 Large Rotations and the Small Strain

The example in section 4.1.2 above illustrated that the small strain approximation is good, provided the deformations are small. However, this is provided also that any *rigid body rotations are small*. To illustrate this, consider a square material element (with sides of unit length) which undergoes a pure rigid body rotation of θ, Fig. 4.1.15. The exact strains 4.1.7 remain zero. The small shear strain remains zero also. However, the small normal strains are seen to be $\varepsilon_{xx} = \varepsilon_{yy} = \cos\theta - 1$. Using a Taylor series expansion, this is equal to $\varepsilon_{xx} = \varepsilon_{yy} \approx -\theta^2/2 + \theta^4/24 - \cdots$. Thus, when θ is small, the rotation-induced strains are of the magnitude/order θ^2. If θ is of the same order as the strains themselves, i.e. in the range $10^{-6} - 10^{-2}$, then θ^2 will be very much smaller than θ and the rotation-induced strains will not introduce any inaccuracy; the small strains will be a good approximation to the actual strains. If, however, the rotation is large, then the engineering normal strains will be wildly inaccurate. For example, when $\theta = 45^{\circ}$, the rotation-induced normal strains are ≈ -0.3, and will likely be larger than the actual strains occurring in the material.

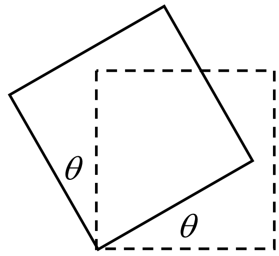


Figure 4.1.15: an element undergoing a rigid body rotation

As an example, consider a cantilevered beam which undergoes large bending, Fig. 4.1.16. The shaded element shown might well undergo small normal and shear strains. However, because of the large rotation of the element, additional spurious engineering normal strains are induced. Use of the precise definition, Eqn. 4.1.7, is required in cases such as this.

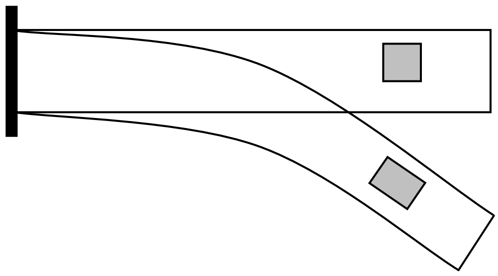


Figure 4.1.16: Large rotations of an element in a bent beam

4.1.6 Three Dimensional Strain

The above can be generalized to three dimensions. In the general case, there are three normal strains, $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}$, and three shear strains, $\varepsilon_{xy}, \varepsilon_{yz}, \varepsilon_{zx}$. The ε_{zz} strain corresponds to a change in length of a line element initially lying along the *z* axis. The ε_{yz} strain corresponds to half the change in the originally right angle of two perpendicular line elements aligned with the *y* and *z* axes, and similarly for the ε_{zx} strain. Straining in the $y-z$ plane ($\varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{yz}$) is illustrated in Fig. 4.1.17 below.

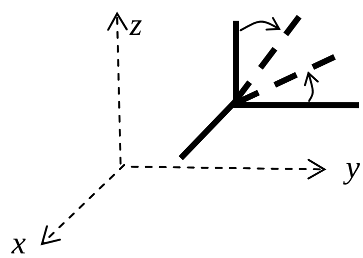


Figure 4.1.17: strains occurring in the $y-z$ plane

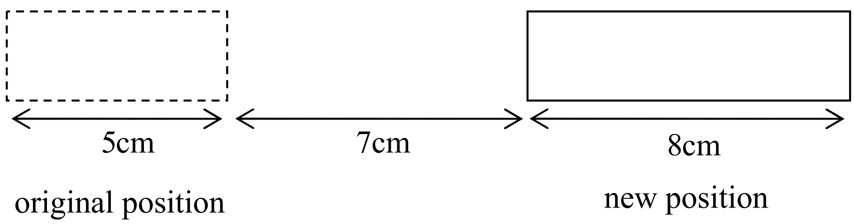
The 3D strains can be represented in the (symmetric) matrix form

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{4.1.11}$$

As with the stress (see Eqn. 3.4.5), there are nine components in 3D, with 6 of them being independent.

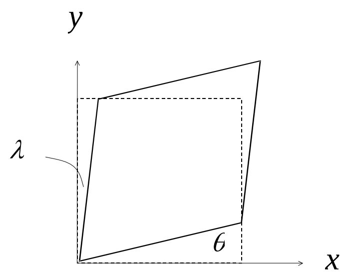
4.1.7 Problems

1. Consider a rod which moves and deforms (uniformly) as shown below.
   (a) What is the displacement of the left-hand end of the rod?
   (b) What is the engineering strain at the left-hand end of the rod



2. A slender rod of initial length 2cm is extended (uniformly) to a length 4cm. It is then compressed to a length of 3cm.
   (a) Calculate the engineering strain and the true strain for the extension
   (b) Calculate the engineering strain and the true strain for the compression
   (c) Calculate the engineering strain and the true strain for one step, i.e. an extension from 2cm to 3cm.
   (d) From your calculations in (a,b,c), which of the strain measures is additive?

3. An element undergoes a homogeneous strain, as shown. There is no normal strain in the element. The angles are given by $\lambda = 0.001$ and $\theta = 0.002$ radians. What is the (tensorial) shear strain in the element?

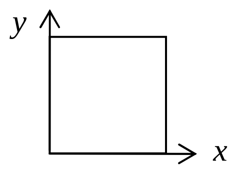


4. In a fixed $x - y$ reference system established for the test of a large component, three points *A*, *B* and *C* on the component have the following coordinates before and after loading (see Figure 4.1.10):

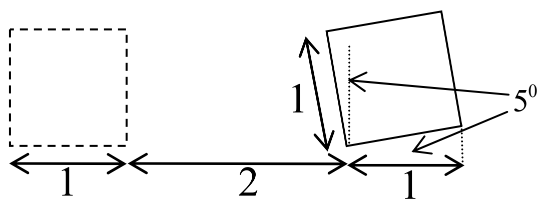
$$C:(0.0000,1.5000) \qquad C':(-0.0025,1.5030)$$
$$A:(0.0000,0.0000) \qquad A':(0.0000,0.0000)$$
$$B:(2.0000,0.0000) \qquad B':(2.0045,0.0000)$$

Determine the actual strains and the small strains (at/near point *A*). What is the error in the small strain compared to the actual strains?

5. Sketch the deformed shape for the material shown below under the following strains (*A*, *B* constant):
   (i) $\varepsilon_{xx} = A > 0$ (taking $\varepsilon_{yy} = \varepsilon_{xy} = 0$) – assume that the right-hand edge is fixed
   (ii) $\varepsilon_{yy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{xy} = 0$) – assume that the lower edge is fixed
   (iii) $\varepsilon_{xy} = B < 0$ (with $\varepsilon_{xx} = \varepsilon_{yy} = 0$) – assume that the left-hand edge is fixed



6. The element shown below undergoes the change in position and dimensions shown (dashed square = undeformed). What are the three engineering strains $\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}$?



4.2 Plane Strain

A state of plane strain is defined as follows:

Plane Strain: If the strain state at a material particle is such that the only non-zero strain components act in one plane only, the particle is said to be in plane strain.

The axes are usually chosen such that the $x-y$ plane is the plane in which the strains are non-zero, Fig. 4.2.1.

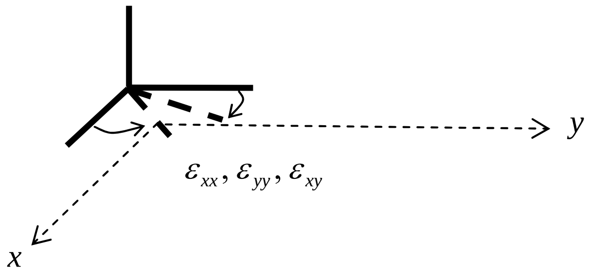


Figure 4.2.1: non-zero strain components acting in the $x-y$ plane

Then $\varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0$. The fully three dimensional strain matrix reduces to a two dimensional one:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix} \tag{4.2.1}$$

4.2.1 Analysis of Plane Strain

Stress transformation formulae, principal stresses, stress invariants and formulae for maximum shear stress were presented in §4.4-§4.5. The strain is very similar to the stress. They are both mathematical objects called tensors, having nine components, and all the formulae for stress hold also for the strain. All the equations in section 3.5.2 are valid again in the case of plane strain, with σ replaced with ε. This will be seen in what follows.

Strain Transformation Formula

Consider two perpendicular line-elements lying in the coordinate directions x and y, and suppose that it is known that the strains are $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$, Fig. 4.2.2. Consider now a second coordinate system, with axes x', y', oriented at angle θ to the first system, and consider line-elements lying along these axes. Using some trigonometry, it can be shown that the line-elements in the second system undergo strains according to the following (two dimensional) **strain transformation equations** (see the Appendix to this section, §4.2.5, for their derivation):

$$\boxed{\begin{aligned}\varepsilon'_{xx} &= \cos^2\theta\varepsilon_{xx} + \sin^2\theta\varepsilon_{yy} + \sin 2\theta\varepsilon_{xy}\\ \varepsilon'_{yy} &= \sin^2\theta\varepsilon_{xx} + \cos^2\theta\varepsilon_{yy} - \sin 2\theta\varepsilon_{xy}\\ \varepsilon'_{xy} &= \sin\theta\cos\theta(\varepsilon_{yy} - \varepsilon_{xx}) + \cos 2\theta\varepsilon_{xy}\end{aligned}}$$

Strain Transformation Formulae (4.2.2)

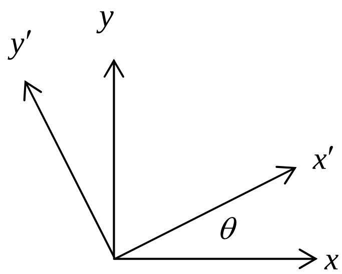


Figure 4.2.2: A rotated coordinate system

Note the similarity between these equations and the stress transformation formulae, Eqns. 3.4.9. Although they have the same structure, the stress transformation equations were derived using Newton's laws, whereas no physical law is used to derive the strain transformation equations 4.2.2, just geometry.

Eqns. 4.2.2 are valid only when the strains are small (as can be seen from their derivation in the Appendix to this section), and the engineering/small strains are assumed in all of which follows. The exact strains, Eqns. 4.1.7, do not satisfy Eqn. 4.2.2 and for this reason they are rarely used – when the strains are large, other strain measures, such as those in Eqns. 4.1.4, are used.

Principal Strains

Using exactly the same arguments as used to derive the expressions for principal stress, there is always at least one set of perpendicular line elements which stretch and/or contract, but which do not undergo angle changes. The strains in this special coordinate system are called **principal strains**, and are given by (compare with Eqns. 3.5.5)

$$\boxed{\begin{aligned}\varepsilon_1 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2}\\ \varepsilon_2 &= \tfrac{1}{2}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) - \sqrt{\tfrac{1}{4}(\varepsilon_{xx} - \varepsilon_{yy})^2 + \varepsilon_{xy}^2}\end{aligned}}$$

Principal Strains (4.2.3)

Further, it can be shown that ε_1 is the maximum normal strain occurring at the point, and that ε_2 is the minimum normal strain occurring at the point.

The **principal directions**, that is, the directions of the line elements which undergo the principal strains, can be obtained from (compare with Eqns. 3.5.4)

$$\tan 2\theta = \frac{2\varepsilon_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \tag{4.2.4}$$

Here, θ is the angle at which the principal directions are oriented with respect to the x axis, Fig. 4.2.2.

Maximum Shear Strain

Analogous to Eqn. 3.5.9, the maximum shear strain occurring at a point is

$$\varepsilon_{xy}\Big|_{\max} = \frac{1}{2}\left(\varepsilon_1 - \varepsilon_2\right) \tag{4.2.5}$$

and the perpendicular line elements undergoing this maximum angle change are oriented at 45° to the principal directions.

Example (of Strain Transformation)

Consider the block of material in Fig. 4.2.3a. Two sets of perpendicular lines are etched on its surface. The block is then stretched, Fig. 4.2.3b.

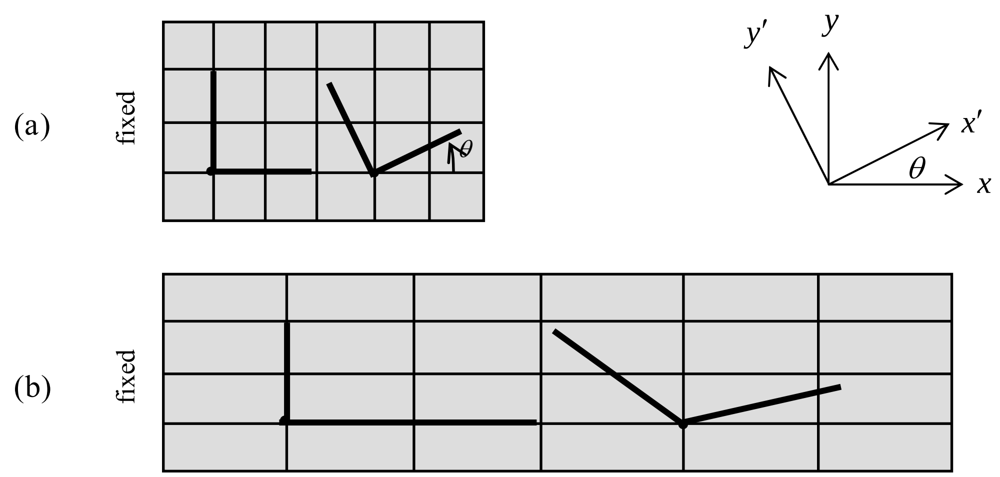


Figure 4.2.3: A block with strain measured in two different coordinate systems

This is a homogeneous deformation, that is, *the strain is the same at all points*. However, in the $x - y$ description, $\varepsilon_{xx} > 0$ and $\varepsilon_{yy} = \varepsilon_{xy} = 0$, but in the $x' - y'$ description, none of the strains is zero. The two sets of strains are related through the strain transformation equations.

■

Example (of Strain Transformation)

As another example, consider a square material element which undergoes a pure shear, as illustrated in Fig. 4.2.4, with

$$\varepsilon_{xx} = \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = 0.01$$

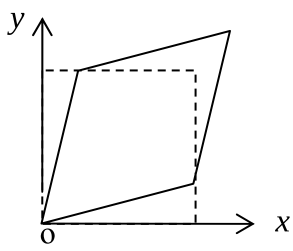


Figure 4.2.4: A block under pure shear

From Eqn. 4.2.3, the principal strains are $\varepsilon_1 = +0.01, \varepsilon_2 = -0.01$ and the principal directions are obtained from Eqn. 4.2.4 as $\theta = \pm 45^\circ$. To find the direction in which the maximum normal strain occurs, put $\theta = +45^\circ$ in the strain transformation formulae to find that $\varepsilon_1 = \varepsilon'_{xx} = +0.01$, so the deformation occurring in a piece of material whose sides are aligned in these principal directions is as shown in Fig. 4.2.5.

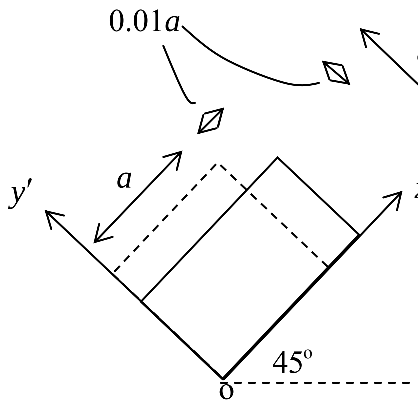


Figure 4.2.5: Principal strains for the block in pure shear

The strain as viewed along the principal directions, and also using the $x - y$ system, are as shown in Fig. 4.2.6.

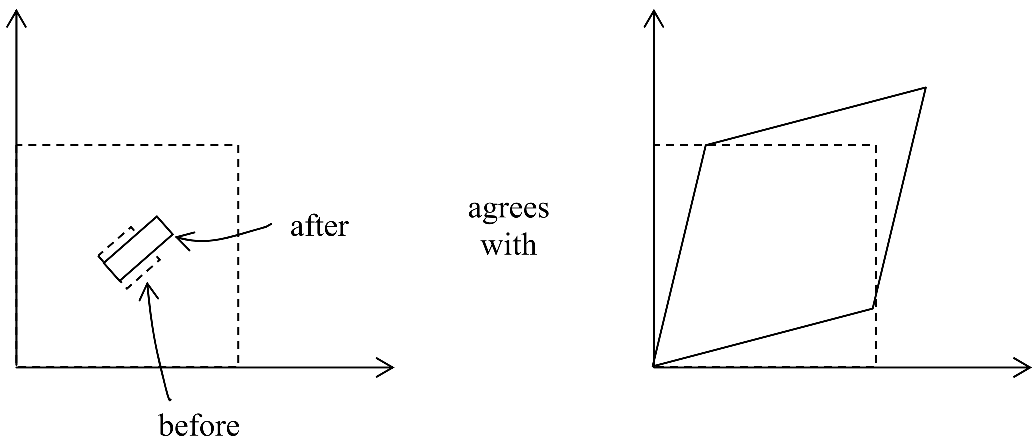


Figure 4.2.6: Strain viewed from two different coordinate systems

The two deformations, square into diamond and square into rectangle, look very different, but they are actually the same thing. For example, the square which deforms into the diamond can be considered to be made up of an infinite number of small rotated squares, Fig. 4.2.7. These then deform into rectangles, which then form the diamond.

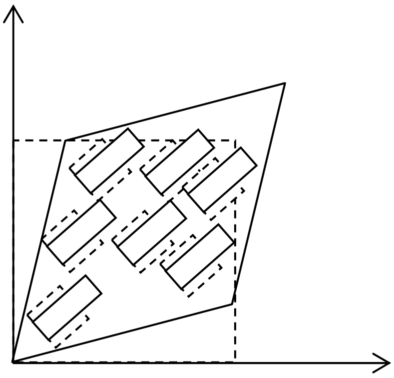


Figure 4.2.7: Alternative viewpoint of the strains in Fig. 4.2.6

Note also that, since the original $x-y$ axes were oriented at $\pm 45°$ to the principal directions, these axes are those of maximum shear strain – the original $\varepsilon_{xy} = 0.01$ is the maximum shear strain occurring at the material particle.

■

4.2.2 Thick Components

It turns out that, just as the state of plane stress often arises in thin components, a state of plane strain often arises in very thick components.

Consider the three dimensional block of material in Fig. 4.2.7. The material is constrained from undergoing normal strain in the z direction, for example by preventing movement with rigid immovable walls – and so $\varepsilon_{zz} = 0$.

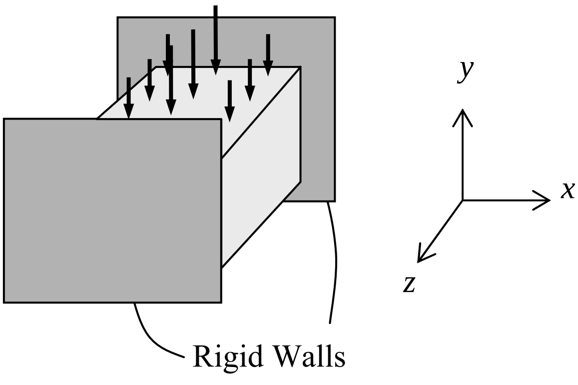


Figure 4.2.7: A block of material constrained by rigid walls

If, in addition, the loading is as shown in Fig. 4.2.7, i.e. it is the same on all cross sections parallel to the $y-z$ plane (or $x-z$ plane) – then the line elements shown in Fig. 4.2.8 will remain perpendicular (although they might move out of plane).

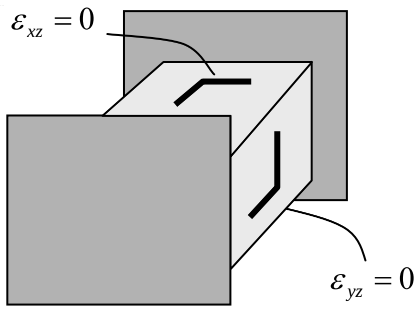


Figure 4.2.8: Line elements etched in a block of material – they remain perpendicular in a state of plane strain

Then $\varepsilon_{xz} = \varepsilon_{yz} = 0$. Thus a state of plane strain will arise.

The problem can now be analysed using the three independent strains, which simplifies matters considerable. Once a solution is found for the deformation of one plane, the solution has been found for the deformation of the whole body, Fig. 4.2.9.

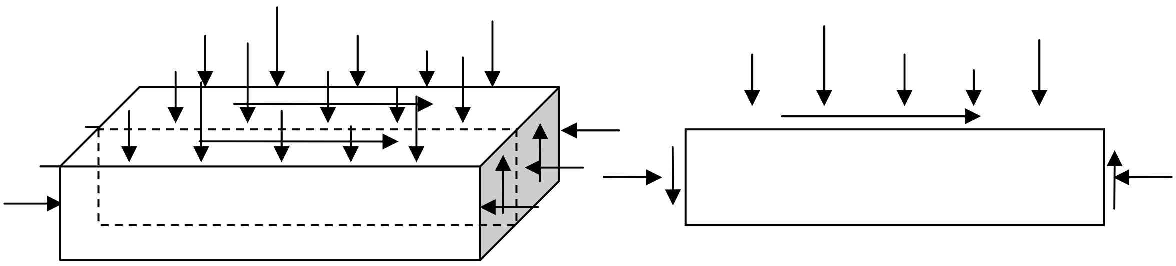


Figure 4.2.9: three dimensional problem reduces to a two dimensional one for the case of plane strain

Note that reaction stresses σ_{zz} act over the ends of the large mass of material, to prevent any movement in the z direction, i.e. ε_{zz} strains, Fig. 4.2.10.

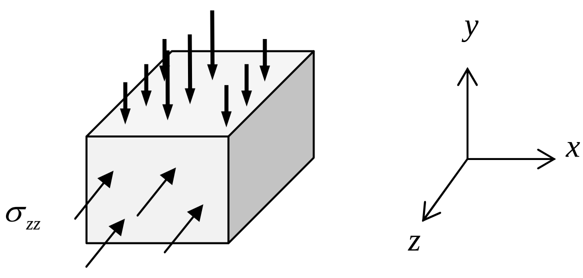


Figure 4.2.10: end-stresses required to prevent material moving in the *z* direction

A state of plane strain will also exist in thick structures without end walls. Material towards the centre is constrained by the mass of material on either side and will be (approximately) in a state of plane strain, Fig. 4.2.10.

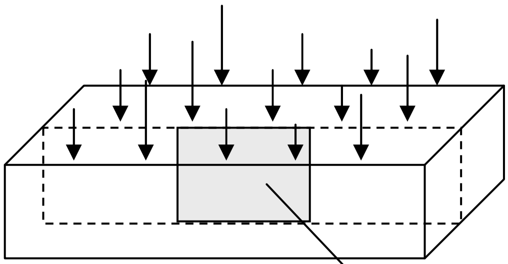


Figure 4.2.10: material in an approximate state of plane strain

Plane Strain is useful when solving many types of problem involving thick components, even when the ends of the mass of material are allowed to move (as in Fig. 4.2.10), using a concept known as **generalised plane strain** (see more advanced mechanics material).

4.2.3 Mohr's Circle for Strain

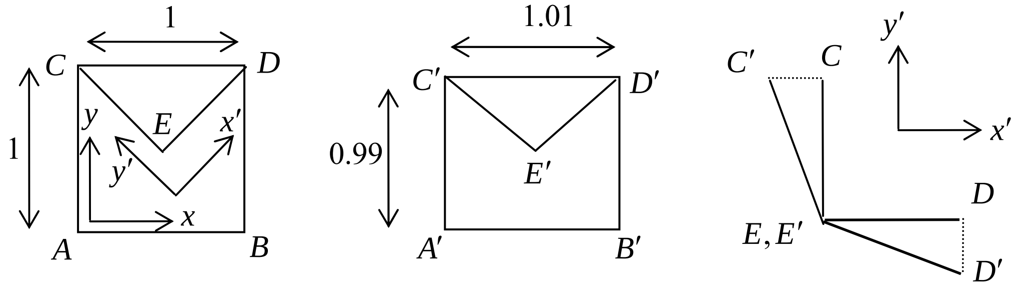
Because of the similarity between the stress transformation equations 3.4.9 and the strain transformation equations 4.2.2, Mohr's Circle for strain is identical to Mohr's Cirlce for stress, section 3.5.5, with σ replaced by ε (and τ replaced by ε_{xy}).

4.2.4 Problems

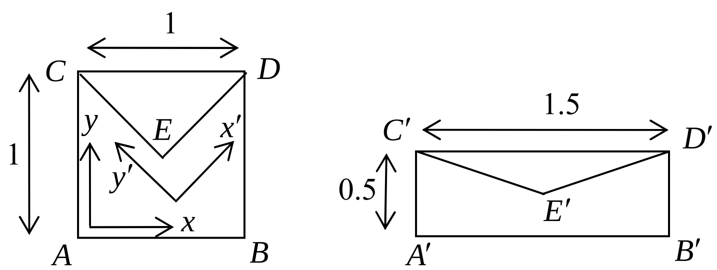
1. In Fig. 4.2.3, take $\theta = 30^{\circ}$ and $\varepsilon_{xx} = 0.02$.
   (a) Calculate the strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$.
   (b) What are the principal strains?
   (c) What is the maximum shear strain?
   (d) Of all the line elements which could be etched in the block, at what angle θ to the x axis are the perpendicular line elements which undergo the largest angle change from the initial right angle?

2. Consider the undeformed rectangular element below left which undergoes a uniform strain as shown centre.
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$. Hint: use the two half-diagonals EC and ED sketched; by superimposing points E, E' (to remove the rigid body motion of E), it will be seen that point D moves straight down and C moves left, when viewed along the x', y' axes, as shown below right.
   (c) Use the strain transformation formulae 4.2.2 and your results from (a) to check your results from (b). Are they the same?

(d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



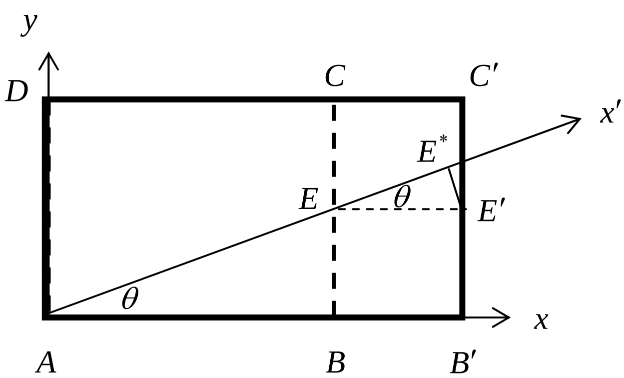
3. Repeat problem 2 only now consider the larger deformation shown below:
   (a) Calculate the engineering strains $\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}$
   (b) Calculate the engineering strains $\varepsilon'_{xx}, \varepsilon'_{yy}, \varepsilon'_{xy}$
   (c) Use the strain transformation formulae 2.4.2 and your results from (a) to check your results from (b). Are they accurate?
   (d) What is the actual unit change in length of the half-diagonals? Does this agree with your result from (b)?



4.2.5 Appendix to §4.2

Derivation of the Strain Transformation Formulae

Consider an element $ABCD$ undergoing a strain ε_{xx} with $\varepsilon_{yy} = \varepsilon_{xy} = 0$ to $AB'C'D$ as shown in the figure below.



In the $x - y$ coordinate system, by definition, $\varepsilon_{xx} = BB' / AB$. In the $x' - y'$ system, AE moves to AE', and one has

$$\varepsilon'_{xx} = \frac{EE^*}{AE} = \frac{\cos\theta\, EE'}{AB / \cos\theta} = \cos^2\theta \frac{BB'}{AB}$$

which is the first term of Eqn. 4.2.2a. Also,

$$\varepsilon'_{xy} = -\frac{E'E^*}{AE} = \frac{\sin\theta\, EE'}{AB / \cos\theta} = -\sin\theta\cos\theta \frac{BB'}{AB}$$

which is in Eqn. 4.2.2c. The remainder of the transformation formulae can be derived in a similar manner.

4.3 Volumetric Strain

The volumetric strain is defined as follows:

Volumetric Strain: The volumetric strain is the unit change in volume, i.e. the change in volume divided by the original volume.

4.3.1 Two-Dimensional Volumetric Strain

Analogous to Eqn 3.5.1, the strain invariants are

$$\boxed{\begin{aligned} I_1 &= \varepsilon_{xx} + \varepsilon_{yy} \\ I_2 &= \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2 \end{aligned}} \qquad \textbf{Strain Invariants} \qquad (4.3.1)$$

Using the strain transformation formulae, Eqns. 4.2.2, it will be verified that these quantities remain unchanged under any rotation of axes.

The first of these has a very significant physical interpretation. Consider the deformation of the material element shown in Fig. 4.3.1a. The volumetric strain is

$$\begin{aligned} \frac{\Delta V}{V} &= \frac{(a+\Delta a)(b+\Delta b) - ab}{ab} \\ &= (1+\varepsilon_{xx})(1+\varepsilon_{yy}) - 1 \\ &= \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy} \end{aligned} \qquad (4.3.2)$$

If the strains are small, the term $\varepsilon_{xx}\varepsilon_{yy}$ will be very much smaller than the other two terms, and the volumetric strain in that case is given by

$$\boxed{\frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy}} \qquad \textbf{Volumetric Strain} \qquad (4.3.3)$$

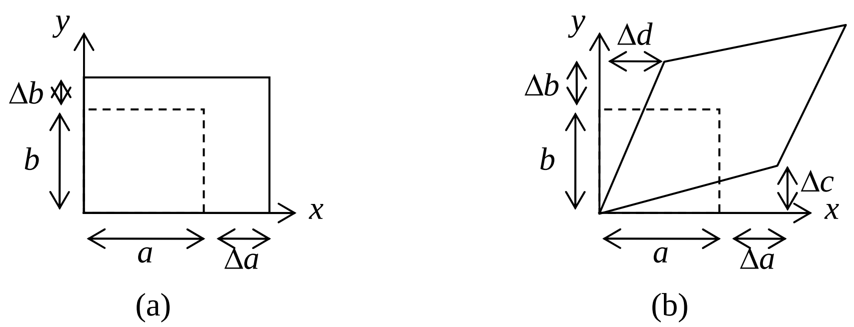


Figure 4.3.1: deformation of a material element; (a) normal deformation, (b) with shearing

Since by Eqn. 4.3.1 the volume change is an invariant, the normal strains in any coordinate system may be used in its evaluation. This makes sense: the volume change cannot depend on the particular axes we choose to measure it. In particular, the principal strains may be used:

$$\frac{\Delta V}{V} = \varepsilon_1 + \varepsilon_2 \tag{4.3.4}$$

The above calculation was carried out for stretching in the *x* and *y* directions, but the result is valid for any arbitrary deformation. For example, for the general deformation shown in Fig. 4.3.1b, some geometry shows that the volumetric strain is $\Delta V / V = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}^2$, which again reduces to Eqns 4.3.3, 4.3.4, for small strains.

An important consequence of Eqn. 4.3.3 is that *normal strains induce volume changes*, whereas *shear strains induce a change of shape but no volume change*.

4.3.2 Three Dimensional Volumetric Strain

A slightly different approach will be taken here in the three dimensional case, so as not to simply repeat what was said above, and to offer some new insight into the concepts.

Consider the element undergoing strains $\varepsilon_{xx}, \varepsilon_{xy}$, etc., Fig. 4.3.2a. The same deformation is viewed along the principal directions in Fig. 4.3.2b, for which only normal strains arise.

The volumetric strain is:

$$\begin{aligned} \frac{\Delta V}{V} &= \frac{(a+\Delta a)(b+\Delta b)(c+\Delta c) - abc}{abc} \\ &= (1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - 1 \\ &\approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \end{aligned} \tag{4.3.5}$$

and the squared and cubed terms can be neglected because of the small-strain assumption.

Since any elemental volume such as that in Fig. 4.3.2a can be constructed out of an infinite number of the elemental cubes shown in Fig. 4.3.3b (as in Fig. 4.2.7), this result holds for any elemental volume irrespective of shape.

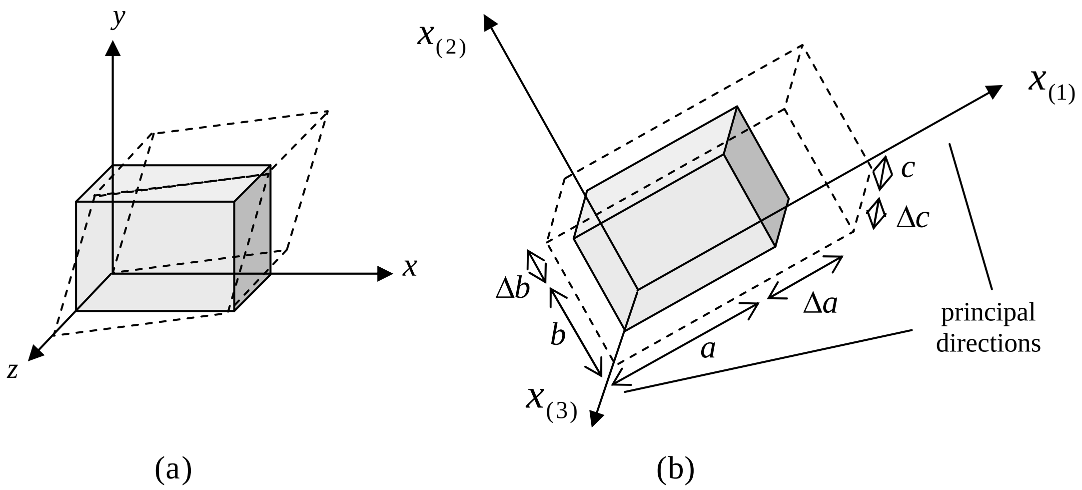


Figure 4.3.2: A block of deforming material; (a) subjected to an arbitrary strain; (a) principal strains