3.4b Stress Transformation: Further Aspects

Here, it will be shown that the Stress Transformation Equations are valid also when (i) there are body forces, (ii) the body is accelerating and (iii) the stress and other quantities are not uniform. We will also examine the fully three-dimensional stress subject to the transformation.

Suppose that a body force $\mathbf{F}_b = (\mathbf{F}_b)_x \mathbf{i} + (\mathbf{F}_b)_y \mathbf{j}$ acts on the material and that the material is accelerating with an acceleration $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j}$. The components of body force and acceleration are shown in Fig. 3.4.10 (a reproduction of Fig. 3.4.8).

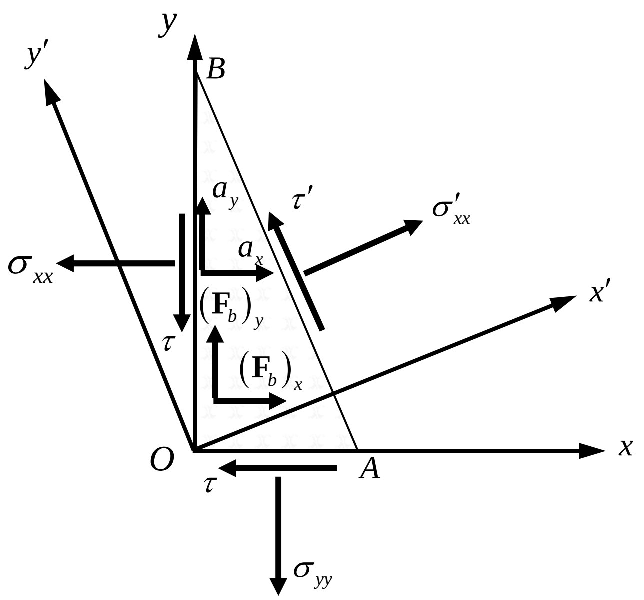

Figure 3.4.10: a free body diagram of a triangular element of material, including a body force and acceleration

The body force will vary depending on the size of the material under consideration, e.g. the force of gravity $\mathbf{F}_b = m\mathbf{g}$ will be larger for larger materials; therefore consider a quantity which is independent of the amount of material: the body force per unit mass, $\mathbf{F}_b / m$. Then, Eqn 3.4.6 now reads

$$\begin{aligned}\sum F_{x'}: \quad & \sigma'_{xx}|AB| - \sigma_{xx}|OB|\cos\theta - \sigma_{yy}|OA|\sin\theta - \tau|OB|\sin\theta - \tau|OA|\cos\theta \\ & + (\mathbf{F}_b/m)_x m\cos\theta + (\mathbf{F}_b/m)m\sin\theta + ma_x\cos\theta + ma_y\sin\theta = 0\end{aligned} \tag{3.4.11}$$

where m is the mass of the triangular portion of material. The volume of the triangle is $\frac{1}{2}|OA||OB| = |AB|^2/\sin 2\theta$ so that, this time, when 3.4.11 is divided through by $|AB|$, one is left with

$$\begin{aligned}\sigma'_{xx} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau\sin 2\theta \\ &\quad - |AB|\rho\left\{(\mathbf{F}_b/m)_x/2\sin\theta + (\mathbf{F}_b/m)/2\cos\theta + a_x/2\sin\theta + a_y/2\cos\theta\right\}\end{aligned} \tag{3.4.12}$$

where ρ is the density. Now, as the element is shrunk in size down to the vertex *O*, $|AB| \to 0$, and Eqn. 3.4.6 is recovered. Thus the Stress Transformation Equations are valid provided the element under consideration is very small; in the limit, they are valid "at the point" *O*.

Finally, consider the case where the stress is not uniform over the faces of the triangular portion of material. Intuitively, it can be seen that, if one again shrinks the portion of material down in size to the vertex *O*, the Stress Transformation Equations will again be valid, with the quantities $\sigma'_{xx}, \sigma_{xx}, \sigma_{yy}$ etc. being the values "at" the vertex. To be more precise, consider the σ_{xx} stress acting over the face $|OB|$ in Fig. 3.4.11. No matter how the stress varies in the material, if the distance $|OB|$ is small, the stress can be approximated by a linear stress distribution, Fig. 3.4.11b. This linear distribution can itself be decomposed into two components, a uniform stress of magnitude σ^o_{xx} (the value of σ_{xx} at the vertex) and a triangular distribution with maximum value $\Delta\sigma_{xx}$. The resultant force on the face is then $|OB|\left(\sigma^o_{xx} + \Delta\sigma_{xx}/2\right)$. This time, as the element is shrunk in size, $\Delta\sigma_{xx} \to 0$ and Eqn. 3.4.6 is again recovered. The same argument can be used to show that the Stress Transformation Equations are valid for any varying stress, body force or acceleration.

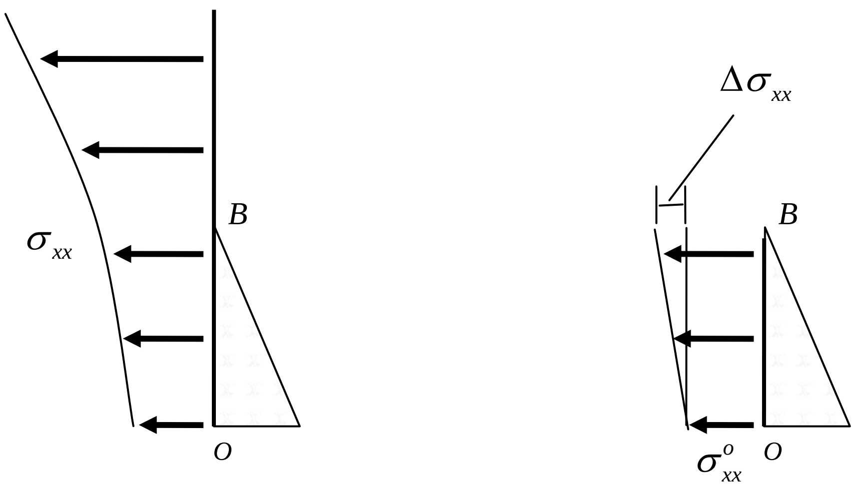

Figure 3.4.11: stress varying over a face; (a) stress is linear over OB if OB is small, (b) linear distribution of stress as a uniform stress and a triangular stress

Three Dimensions Re-visited

As the planes were rotated in the two-dimensional analysis, no consideration was given to the stresses acting in the "third dimension". Considering again a three dimensional block, Fig. 3.4.12, there is only one traction vector acting *on* the $x-y$ plane at the material particle, **t**. This traction vector can be described in terms of the x, y, z axes as $\mathbf{t} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$, Fig 3.4.12a. Alternatively, it can be described in terms of the x', y', z' axes as $\mathbf{t} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}' + \sigma'_{zz}\mathbf{k}'$, Fig 3.4.12b.

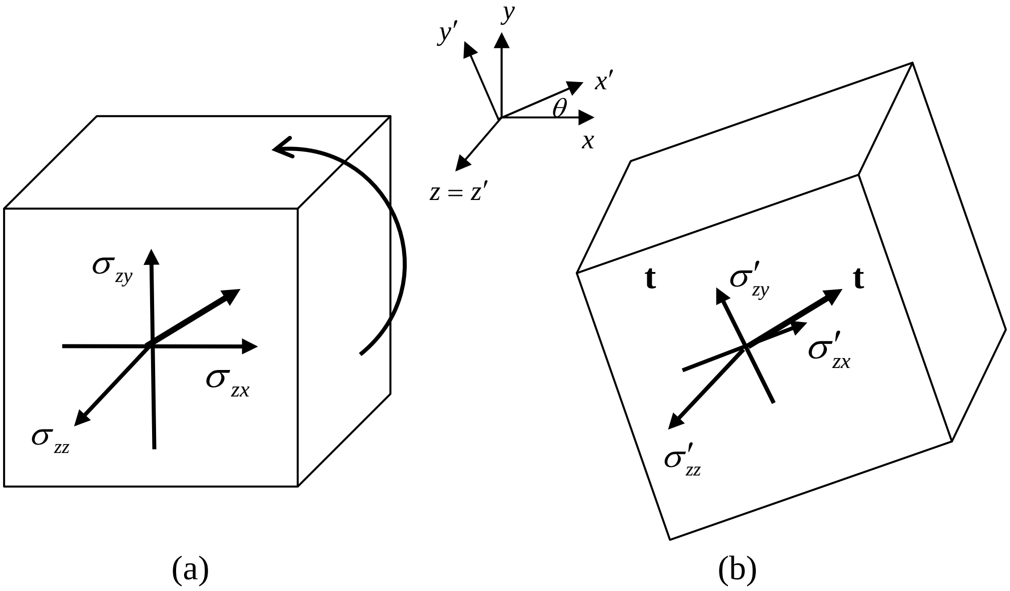

Figure 3.4.12: a three dimensional material element; (a) original element, (b) rotated element (rotation about the *z* axis)

With the rotation only happening *in* the $x-y$ plane, about the *z* axis, one has $\sigma_{zz} = \sigma'_{zz}$, $\mathbf{k} = \mathbf{k}'$. One can thus examine the two dimensional $x-y$ plane shown in Fig. 3.4.13, with

$$\sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} = \sigma'_{zx}\mathbf{i}' + \sigma'_{zy}\mathbf{j}'. \tag{3.4.13}$$

Using some trigonometry, one can see that

$$\begin{aligned} \sigma'_{zx} &= +\sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta \\ \sigma'_{zy} &= -\sigma_{zx}\sin\theta + \sigma_{zy}\cos\theta \end{aligned}. \tag{3.4.14}$$

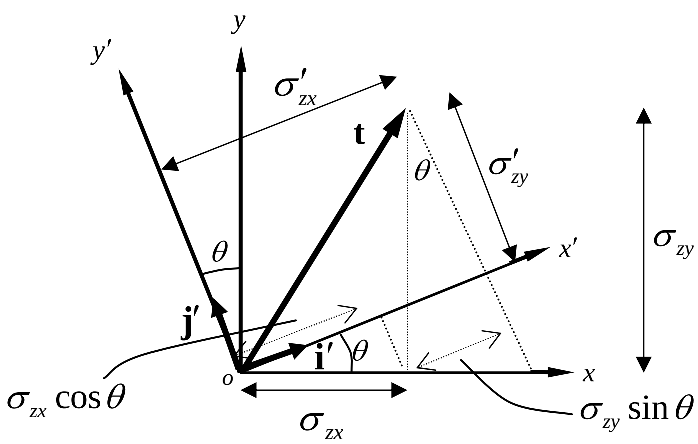

Figure 3.4.12: the traction vector represented using two different coordinate systems