3.3 Internal Stress

The idea of stress considered in §3.1 is not difficult to conceptualise since objects interacting with other objects are encountered all around us. A more difficult concept is the idea of forces and stresses acting *inside* a material, "within the interior where neither eye nor experiment can reach" as Euler put it. It took many great minds working for centuries on this question to arrive at the concept of stress we use today, an idea finally brought to us by Augustin Cauchy, who presented a paper on the subject to the Academy of Sciences in Paris, in 1822.

Augustin Cauchy

3.3.1 Cauchy's Concept of Stress

Uniform Internal Stress

Consider first a long slender block of material subject to equilibrating forces *F* at its ends, Fig. 3.3.1a. If the complete block is in equilibrium, then any sub-division of the block must be in equilibrium also. By imagining the block to be cut in two, and considering free-body diagrams of each half, as in Fig. 3.3.1b, one can see that forces *F* must be acting *within* the block so that each half is in equilibrium. Thus *external loads create internal forces*; internal forces represent the action of one part of a material on another part of the same material across an internal surface. We can take it that a uniform stress $\sigma = F / A$ acts over this interior surface, Fig. 3.3.1b.
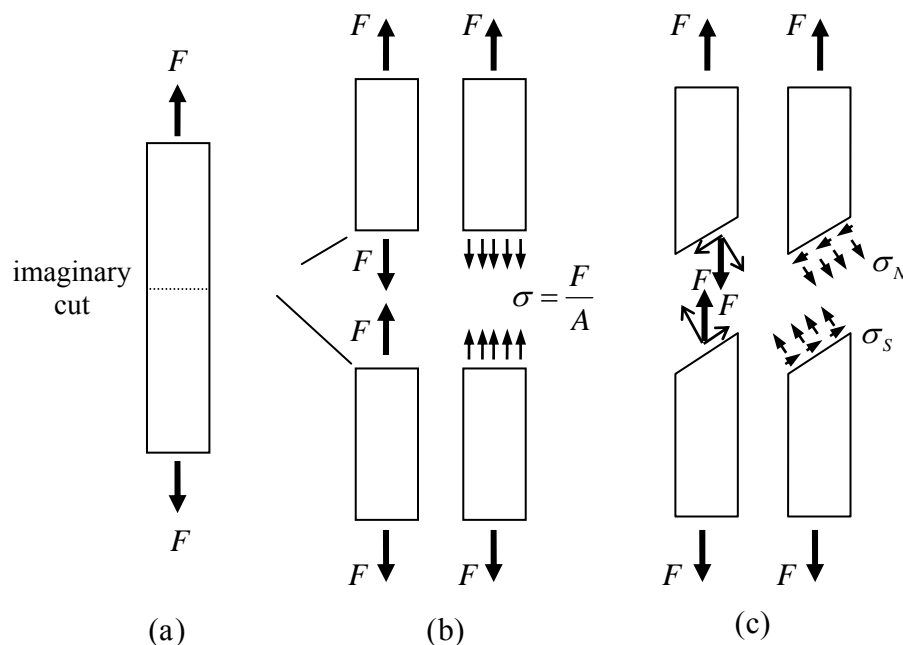

Figure 3.3.1: a slender block of material; (a) under the action of external forces F, (b) internal normal stress σ, (c) internal normal and shear stress

Note that, if the internal forces were not acting over the internal surfaces, the two half-blocks of Fig. 3.3.1b would fly apart; one can thus regard the internal forces as those required to maintain material in an un-cut state.

If the internal surface is at an incline, as in Fig. 3.3.1c, then the internal force required for equilibrium will not act normal to the surface. There will be components of the force normal and tangential to the surface, and thus both normal (σ_N) and shear (σ_S) stresses must arise. Thus, even though the material is subjected to a purely normal load, internal shear stresses develop.

From Fig. 3.3.2a, with the stress given by force divided by area, the normal and shear stresses arising on an interior surface inclined at angle θ to the horizontal are {▲Problem 1}

$$\sigma_N = \frac{F}{A}\cos^2\theta, \qquad \sigma_S = \frac{F}{A}\sin\theta\cos\theta \qquad\qquad (3.3.1)$$

(a)   (b)
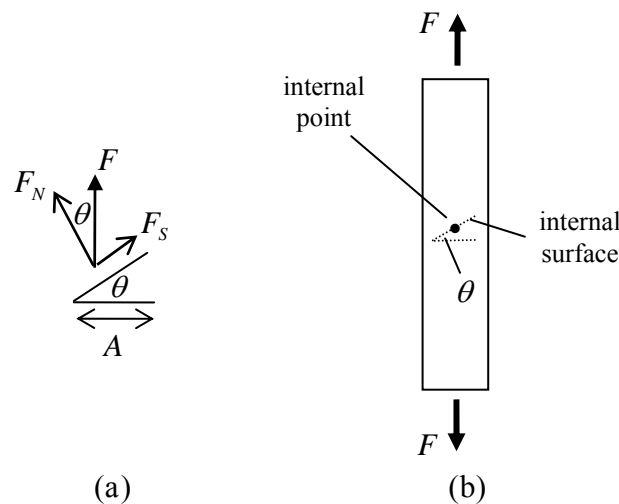
Figure 3.3.2: stress on inclined surface; (a) decomposing the force into normal and shear forces, (b) stress at an internal point

Although stress is associated with surfaces, one can speak of the stress "at a point". For example, consider some point interior to the block, Fig 3.3.2b. The stress there evidently depends on which surface through that point is under consideration. From Eqn. 3.3.1a, the normal stress at the point is a maximum F/A when $\theta = 0$ and a minimum of zero when $\theta = 90°$. The maximum normal stress arising at a point within a material is of special significance, for example it is this stress value which often determines whether a material will fail ("break") there. It has a special name: the **maximum principal stress**. From Eqn. 3.3.1b, the **maximum shear stress** at the point is $\pm F/2A$ and arises on surfaces inclined at $\pm 45°$.

Non-Uniform Internal Stress

Consider a more complex geometry under a more complex loading, as in Fig. 3.3.3. Again, using equilibrium arguments, there will be some stress distribution acting over any

given internal surface. To evaluate these stresses is not a straightforward matter, suffice to say here that they will invariably be *non-uniform* over a surface, that is, the stress at some particle will differ from the stress at a neighbouring particle.
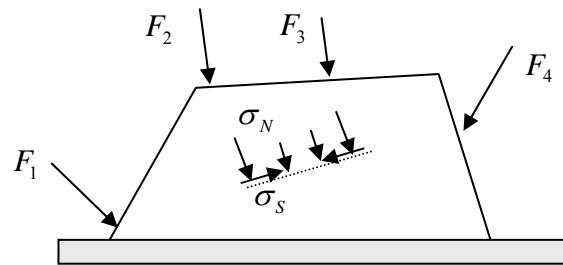
Figure 3.3.3: a component subjected to a complex loading, giving rise to a non-uniform stress distribution over an internal surface

Traction and the Physical Meaning of Internal Stress

All materials have a complex molecular microstructure and each molecule exerts a force on each of its neighbours. The complex interaction of countless molecular forces maintains a body in equilibrium in its unstressed state. When the body is disturbed and deformed into a new equilibrium position, net forces act, Fig. 3.3.4a. An imaginary plane can be drawn through the material, Fig. 3.3.4b. Unlike some of his predecessors, who attempted the extremely difficult task of accounting for all the molecular forces, Cauchy discounted the molecular structure of matter and simply replaced the imagined molecular forces acting on the plane by a single force **F**, Fig 3.3.4c. This is the force exerted by the molecules above the plane *on* the material below the plane and can be attractive or repulsive. Different planes can be taken through the *same* portion of material and, in general, a *different* force will act on the plane, Fig 3.3.4d.


(a)   (b)   (c)   (d)
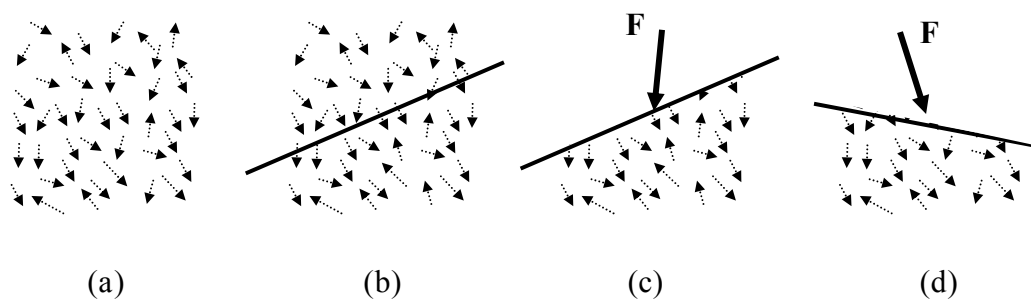
Figure 3.3.4: a multitude of molecular forces represented by a single force; (a) molecular forces, a plane drawn through the material, replacing the molecular forces with an equivalent force F, a different equivalent force F acts on a different plane through the same material

The definition of stress will now be made more precise. First, define the **traction** at some particular point in a material as follows: take a plane of surface area S through the point, on which acts a force F. Next shrink the plane – as it shrinks in size both S and F get smaller, and the direction in which the force acts may change, but eventually the ratio F/S will remain constant and the force will act in a particular direction, Fig. 3.3.5. The
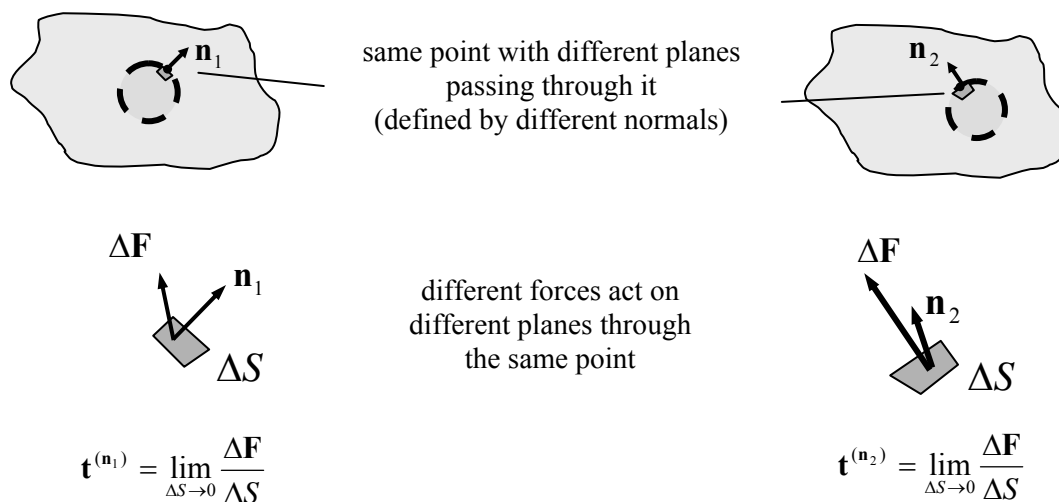
limiting value of this ratio of force over surface area is defined as the **traction vector** (or **stress vector**) **t**:[1]

$$\mathbf{t} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} \qquad (3.3.2)$$
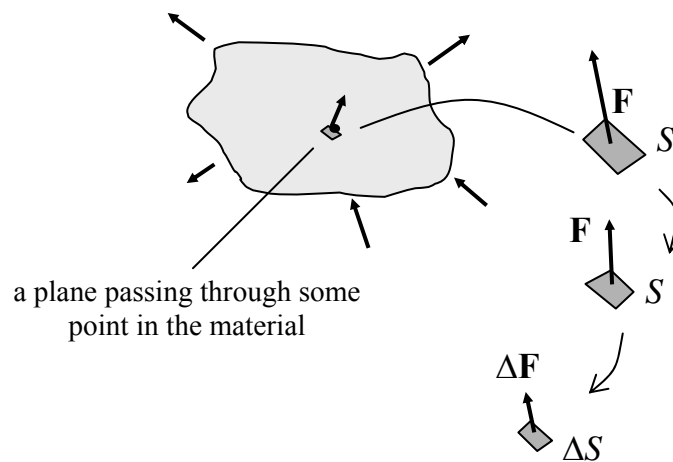
Figure 3.3.5: the traction vector - the limiting value of force over area, as the surface area of the element on which the force acts is shrunk

An infinite number of traction vectors act at any single point, since an infinite number of different planes pass through a point. Thus the notation $\lim_{\Delta S \to 0} \Delta F / \Delta S$ is ambiguous. For this reason the plane on which the traction vector acts must be specified; this can be done by specifying the normal **n** to the surface on which the traction acts, Fig 3.3.6. The traction is thus a special vector – associated with it is not only the direction in which it acts but also a second direction, the normal to the plane upon which it acts.


$$\mathbf{t}^{(\mathbf{n}_1)} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S} \qquad\qquad \mathbf{t}^{(\mathbf{n}_2)} = \lim_{\Delta S \to 0} \frac{\Delta F}{\Delta S}$$

Figure 3.3.6: two different traction vectors acting at the same point

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[1] this does not mean that the force is acting on a surface of zero area – the meaning of this limit is further examined in section 5.4, in the context of the continuum
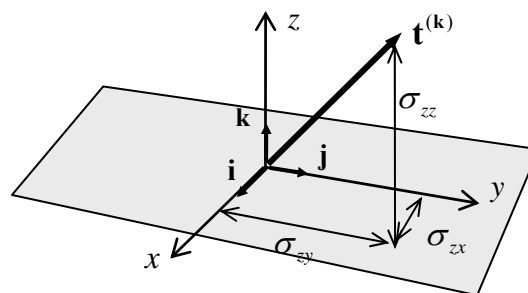
Stress Components

The traction vector can be decomposed into components which act normal and parallel to the surface upon which it acts. These components are called the **stress components**, or simply **stresses**, and are denoted by the symbol σ; subscripts are added to signify the surface on which the stresses act and the directions in which the stresses act.

Consider a particular traction vector acting on a surface element. Introduce a Cartesian coordinate system with base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ so that one of the base vectors is a normal to the surface, and the origin of the coordinate system is positioned at the point at which the traction acts. For example, in Fig. 3.3.7, the $\mathbf{k}$ direction is taken to be normal to the plane, and $\mathbf{t}^{(\mathbf{k})} = t_x\mathbf{i} + t_y\mathbf{j} + t_z\mathbf{k}$.
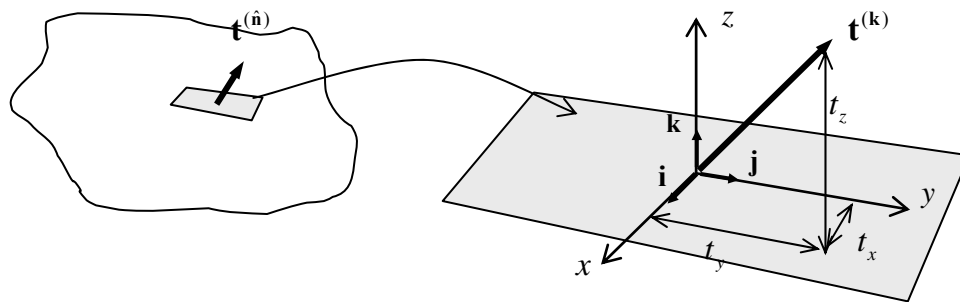
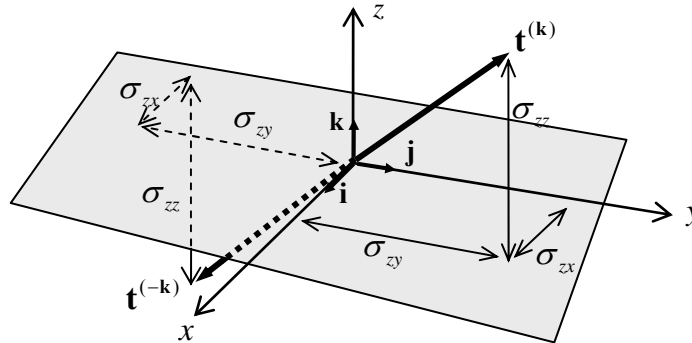**Figure 3.3.7: the components of the traction vector**

Each of these components t_i is represented by σ_{ij} where the first subscript denotes the *direction of the normal* to the plane and the second denotes the *direction of the component*. Thus, re-drawing Fig. 3.3.7 as Fig. 3.3.8: $\mathbf{t}^{(\mathbf{k})} = \sigma_{zx}\mathbf{i} + \sigma_{zy}\mathbf{j} + \sigma_{zz}\mathbf{k}$. The first two stresses, the components acting tangential to the surface, are shear stresses, whereas σ_{zz}, acting normal to the plane, is a normal stress[2].

Figure 3.3.8: stress components – the components of the traction vector

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[2] this convention for the subscripts is not universally followed. Many authors, particularly in the mathematical community, use the exact opposite convention, the first subscript to denote the direction and the second to denote the normal. It turns out that *both conventions are equivalent*, since, as will be shown later, $\sigma_{ij} = \sigma_{ji}$

The traction vector shown in Figs. 3.3.7, 3.3.8, represents the force (per unit area) exerted by the material above the surface *on* the material below the surface. By Newton's third law, an equal and opposite traction must be exerted by the material below the surface on the material above the surface, as shown in Fig. 3.3.9 (thick dotted line). If $\mathbf{t}^{(\mathbf{k})}$ has stress components $\sigma_{zx}, \sigma_{zy}, \sigma_{zz}$, then so should $\mathbf{t}^{(-\mathbf{k})}$: $\mathbf{t}^{(-\mathbf{k})} = \sigma_{zx}(-\mathbf{i}) + \sigma_{zy}(-\mathbf{j}) + \sigma_{zz}(-\mathbf{k}) = -\mathbf{t}^{(\mathbf{k})}$.

Figure 3.3.9: equal and opposite traction vectors – each with the same stress components

Sign Convention for Stress Components

The following convention is used:

> The stress is *positive* when the direction of the normal *and* the direction of the stress component are both positive *or* both negative
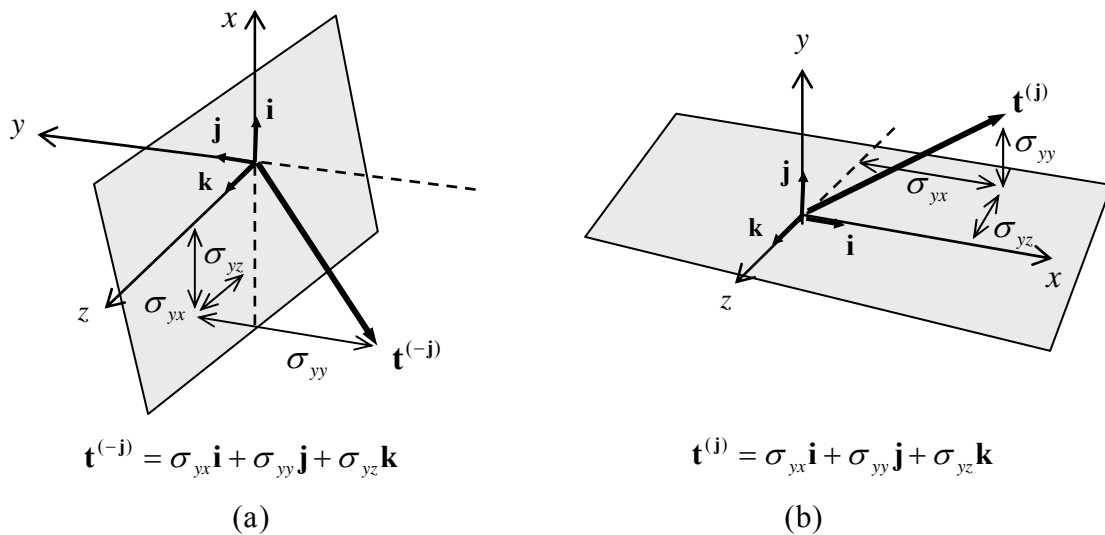> The stress is *negative* when one of the directions is positive and the other is negative

According to this convention, the three stresses in Figs. 3.3.7-9 are all positive.

Looking at the two-dimensional case for ease of visualisation, the (positive and negative) normal stresses and shear stresses on either side of a surface are as shown in Fig. 3.3.10. To clarify this, consider the σ_{yy} stress in Fig. 3.310a: "above" the plane, the normal to the plane is in the positive *y* direction (up) and the component σ_{yy} acts in the positive direction (up), so this stress is positive; "below" the plane, the normal to the plane is in the negative *y* direction (down) and the component σ_{yy} acts in the negative direction (down), so this stress is positive. The simple consequence of this sign convention is that normal stresses which "pull" (tension) are positive and normal stresses which "push" (compression) are negative. Note that the shear stresses always go in opposite directions.
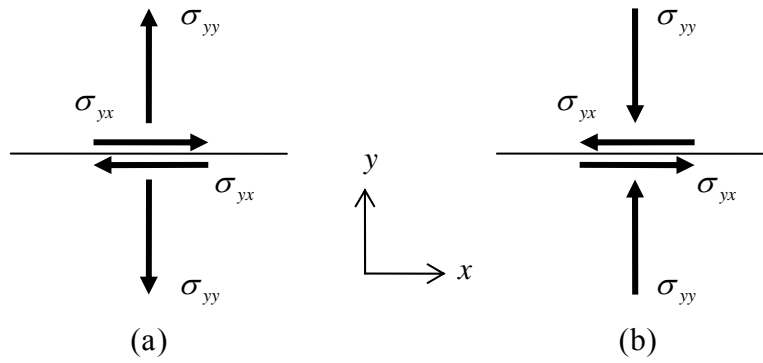
Figure 3.3.10: stresses acting on either side of a material surface: (a) positive stresses, (b) negative stresses

Examples of negative stresses are shown in Fig. 3.3.11 {▲Problem 4}.

$$\mathbf{t}^{(-\mathbf{j})} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$$

(a)

$$\mathbf{t}^{(\mathbf{j})} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$$

(b)

Figure 3.3.11: examples of negative stress components

3.3.2 Real Problems and Saint-Venant's Principle

Some examples have been given earlier of external forces acting on materials. In reality, an external force will be applied to a real material component in a complex way. For example, suppose that a block of material, welded to a large object at one end, is pulled at its other end by a rope attached to a metal hoop, which is itself attached to the block by a number of bolts, Fig. 3.3.12a. The block can be idealised as in Fig 3.3.12b; here, the precise details of the region in which the external force is applied are neglected.
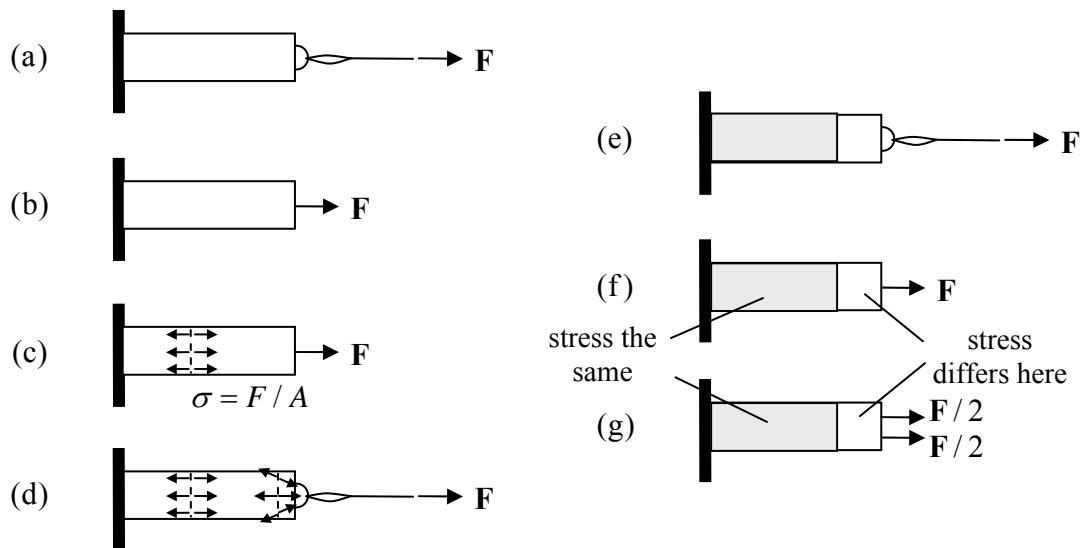
Figure 3.3.12: a block subjected to an external force: (a) real case, (b) ideal model, (c) stress in ideal model, (d) stress in actual material, (e) the stress in the real material, away from the right hand end, is modelled well by either (f) or (g)

According to the earlier discussion, the stress in the ideal model is as in Fig. 3.3.12c. One will find that, in the *real* material, the stress is indeed (approximately) as predicted, but only at an appreciable distance from the right hand end. Near where the rope is attached, the force will differ considerably, as sketched in Fig.3.3.12d.

Thus the ideal models of the type discussed in this section, and in much of this book, are useful only in predicting the stress field in real components in regions away from points of application of loads. This does not present too much of a problem, since the stresses internal to a structure in such regions are often of most interest. If one wants to know what happens near the bolted connection, then one will have to create a complex model incorporating all the details and the problem will be more difficult to solve.

That said, it is an *experimental* fact that if two different force systems are applied to a material, but they are *equivalent force systems*, as in Fig. 3.3.12(f,g), then the stress fields in regions away from where the loads are applied will be the same. This is known as **Saint-Venant's Principle**. Typically, one needs to move a distance away from where the loads are applied roughly equal to the distance over which the loads are applied.

Saint-Venant's principle is extremely important in practical applications: we can replace a complicated problem by a simple model problem; the solution to this latter problem will often give us the information we require.

3.3.3   Problems

1. Derive Eqns. 3.3.1.

2. The four sides of a square block are subjected to equal *forces S*, as illustrated. The length of each side is *l* and the block has unit depth (into the page). What normal and shear *stresses* act along the (dotted) diagonal? [Hint: draw a free body diagram of the upper left hand triangle.]
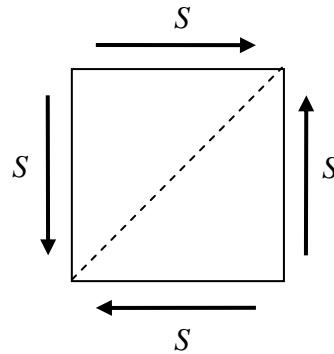

3. A shaft is concreted firmly into the ground. A thick steel rope is looped around the shaft and a force is applied normal to the shaft, as shown. The shaft is in static equilibrium. Draw a free body diagram of the shaft (from the top down to ground level) showing the forces/moments acting on the shaft (including the reaction forces at the ground-level; ignore the weight of the shaft). Draw a free body diagram of the section of shaft from the top down to the cross section at A. Draw a free body diagram of the section of shaft from the top to the cross section at B. Roughly sketch the stresses acting over the (horizontal) internal surfaces of the shaft at A and B.
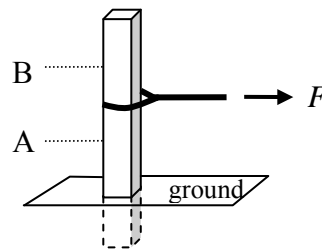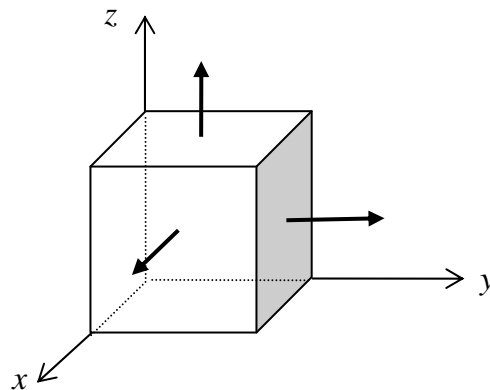

4. In Fig. 3.3.11, which of the stress components is/are negative?

5. Label the following stress component acting on an internal material surface. Is it a positive or negative stress?
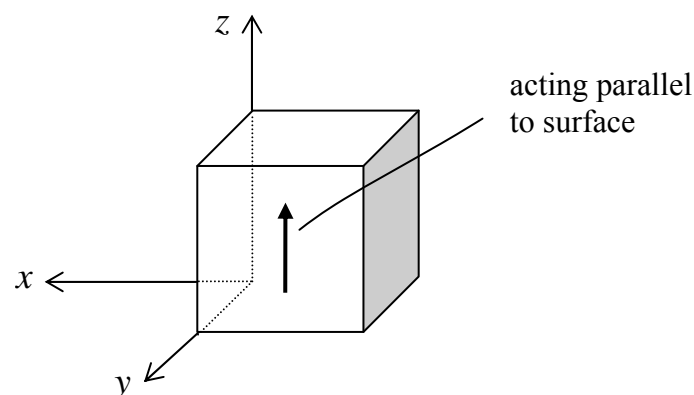
6. Label the following shear stresses. Are they positive or negative?
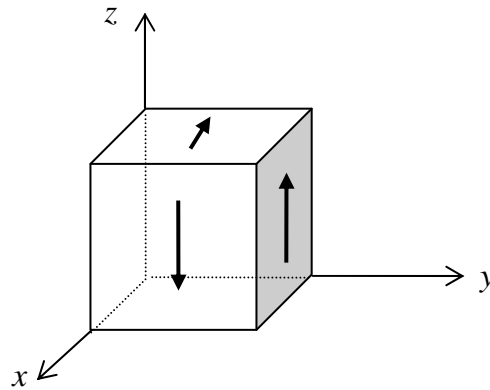
7. Label the following normal stresses. Are they positive or negative?

8. By the definition of the traction vector **t** which acts on the $x - z$ plane,
   $\mathbf{t}^{(j)} = \sigma_{yx}\mathbf{i} + \sigma_{yy}\mathbf{j} + \sigma_{yz}\mathbf{k}$. Sketch these three stress components on the figure below.