9 2D (Plane) Linear Elasticity

9.1 Governing Equations

The equations governing the elasticity problem in the plane are

(1) the two dimensional equations of equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}+b_x=0, \qquad \frac{\partial \sigma_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+b_y=0 \tag{9.1}$$

where σ_{ij} are the stress components and $\mathbf{b}=[b_x, b_y]^{\mathrm{T}}$ is the body force.

(2) the kinematic (strain-displacement) relations

$$\varepsilon_{xx}=\frac{\partial u}{\partial x}, \quad \varepsilon_{yy}=\frac{\partial v}{\partial y}, \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \tag{9.2}$$

where u, v are the displacements in the x and y directions respectively, ε_{ij} are the strain components, and $\gamma_{xy}=2e_{xy}$ is the engineering shear strain.

(3) the constitutive equations (Hooke's Law). These can be either for (assuming the material to be isotropic)

(a) plane stress conditions

$$\begin{aligned}\sigma_{xx}&=\frac{E}{1-\nu^2}\left[\varepsilon_{xx}+\nu\varepsilon_{yy}\right]\\ \sigma_{yy}&=\frac{E}{1-\nu^2}\left[\nu\varepsilon_{xx}+\varepsilon_{yy}\right]\\ \sigma_{xy}&=\frac{E}{2(1+\nu)}\gamma_{xy}\end{aligned} \tag{9.3a}$$

or

(b) plane strain conditions

$$
\begin{aligned}
\sigma_{xx} &= \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}\right] \\
\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_{xx} + (1-\nu)\varepsilon_{yy}\right] \\
\sigma_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy}
\end{aligned}
\tag{9.3b}
$$

where E is Young's modulus and ν is the Poisson's ratio.

The above are eight equations in the eight unknowns of stress (3), strain (3) and displacement (2).

The governing equations are solved subject to the boundary conditions: these are either on displacement (geometric BCs) or on stress (traction BCs)

Note that equations (9.3b) can be obtained from equations (9.3a) by making the following change of material parameters:

$$
E \to \frac{E}{1-\nu^2}, \quad \nu \to \frac{\nu}{1-\nu} \tag{9.4}
$$

Thus if one solves the plane stress problem, one can obtain the plane strain solution by making the substitutions (9.4), and vice versa.

9.1.1 Compatibility

In addition to the above equations, the condition of compatibility must be satisfied:

$$
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} - 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 0 \tag{9.5}
$$

This ensures that the strains can be integrated to give a unique continuous displacement field.

9.1.2 Matrix Notation

Introduce now the following matrix notation for the stresses, strains and constitutive relation:

$$\{\boldsymbol{\sigma}\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}, \qquad \{\mathbf{e}\} = \begin{Bmatrix} e_{xx} \\ e_{yy} \\ \gamma_{xy} \end{Bmatrix}, \qquad \{\boldsymbol{\sigma}\} = [\mathbf{C}]\{\mathbf{e}\} \tag{9.6}$$

where the material property matrix $[\mathbf{C}]$ is

$$[\mathbf{C}] = \begin{cases} \dfrac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-\nu) \end{bmatrix} & \cdots \quad \text{plane}\,\sigma \\[2ex] \dfrac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} & \cdots \quad \text{plane}\,\varepsilon \end{cases} \tag{9.7}$$

Materials which are not isotropic will have a different material property matrix $[\mathbf{C}]$, but the rest of the equations, and the theory which follows, are the same as for the isotropic case.

9.2 Derivation of the FEM Equations

9.2.1 The Weighted Residual Equations

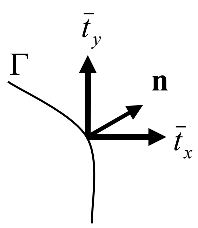
The weighted residual method is applied to the equilibrium equations:

$$\begin{aligned} &\int_\Omega \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right) \omega_1 dS + \int_\Omega b_x \omega_1 dS = 0 \\ &\int_\Omega \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \right) \omega_2 dS + \int_\Omega b_y \omega_2 dS = 0 \end{aligned} \tag{9.8}$$

where Ω is the complete 2D domain. Thus in the FEM, the equilibrium equations are satisfied in an average way over the domain. Green's theorem now leads to

$$
\begin{aligned}
&-\int_{\Omega}\left(\frac{\partial\omega_1}{\partial x}\sigma_{xx}+\frac{\partial\omega_1}{\partial y}\sigma_{xy}\right)dS+\int_{\Gamma}\omega_1\left(\sigma_{xx}n_x+\sigma_{xy}n_y\right)dC+\int_{\Omega}b_x\omega_1 dS=0\\
&-\int_{\Omega}\left(\frac{\partial\omega_2}{\partial x}\sigma_{xy}+\frac{\partial\omega_2}{\partial y}\sigma_{yy}\right)dS+\int_{\Gamma}\omega_2\left(\sigma_{xy}n_x+\sigma_{yy}n_y\right)dC+\int_{\Omega}b_y\omega_2 dS=0
\end{aligned}
\tag{9.9}
$$

where Γ is the boundary of the domain.



Introduce the traction vector $\mathbf{t}=[t_x, t_y]^{\mathrm{T}}$ such that

$$
\begin{aligned}
t_x &= \sigma_{xx}n_x+\sigma_{xy}n_y\\
t_y &= \sigma_{xy}n_x+\sigma_{yy}n_y
\end{aligned}
\tag{9.10}
$$

Thus

$$
\begin{aligned}
&-\int_{\Omega}\left(\frac{\partial\omega_1}{\partial x}\sigma_{xx}+\frac{\partial\omega_1}{\partial y}\sigma_{xy}\right)dS+\int_{\Gamma}\omega_1\bar{t}_x dC+\int_{\Omega}b_x\omega_1 dS=0\\
&-\int_{\Omega}\left(\frac{\partial\omega_2}{\partial x}\sigma_{xy}+\frac{\partial\omega_2}{\partial y}\sigma_{yy}\right)dS+\int_{\Gamma}\omega_2\bar{t}_y dC+\int_{\Omega}b_y\omega_2 dS=0
\end{aligned}
\tag{9.11}
$$

and $\bar{t}_x, \bar{t}_y$ are the tractions on the surface of the material.

Writing in matrix form and substituting in the constitutive relations now leads to

$$
\int_{\Omega}\begin{bmatrix}\frac{\partial\omega_1}{\partial x} & 0 & \frac{\partial\omega_1}{\partial y}\\ 0 & \frac{\partial\omega_2}{\partial y} & \frac{\partial\omega_2}{\partial x}\end{bmatrix}[\mathbf{C}]\{\boldsymbol{\varepsilon}\}dS=\int_{\Omega}\begin{bmatrix}\omega_1 & 0\\ 0 & \omega_2\end{bmatrix}\begin{Bmatrix}b_x\\ b_y\end{Bmatrix}dS+\int_{\Gamma}\begin{bmatrix}\omega_1 & 0\\ 0 & \omega_2\end{bmatrix}\begin{Bmatrix}\bar{t}_x\\ \bar{t}_y\end{Bmatrix}dC \tag{9.12}
$$

The displacements *u* and *v* can be interpolated with the same shape functions N_i of *m* degrees of freedom:

$$
u(x,y)=\sum_{i=1}^{m}N_i(x,y)u_i,\qquad v(x,y)=\sum_{i=1}^{m}N_i(x,y)v_i \tag{9.13}
$$

These interpolations can be written in terms of a shape function matrix $[\mathbf{N}]$,

$$\{\mathbf{u}\} = [\mathbf{N}]\{\mathbf{d}\}, \qquad \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_m & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_m \\ v_m \end{Bmatrix} \tag{9.14}$$

and the strains can then be written in terms of a strain-displacement matrix $[\mathbf{B}]$,

$$\{\boldsymbol{\varepsilon}\} = [\mathbf{B}]\{\mathbf{d}\}, \quad \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \cdots & \dfrac{\partial N_m}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & \cdots & 0 & \dfrac{\partial N_m}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \cdots & \dfrac{\partial N_m}{\partial y} & \dfrac{\partial N_m}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_m \\ v_m \end{Bmatrix} \tag{9.15}$$

Substituting back into (9.12), with the integrations now over an element, gives

$$\left[\int_{\Omega^{(e)}} \begin{bmatrix} \dfrac{\partial \omega_1}{\partial x} & 0 & \dfrac{\partial \omega_1}{\partial y} \\ 0 & \dfrac{\partial \omega_2}{\partial y} & \dfrac{\partial \omega_2}{\partial x} \end{bmatrix} [\mathbf{C}][\mathbf{B}]\, dS \right] \{\mathbf{d}\} = \int_{\Omega^{(e)}} \begin{Bmatrix} b_x \omega_1 \\ b_y \omega_2 \end{Bmatrix} dS + \int_{\Gamma^{(e)}} \begin{Bmatrix} \bar{t}_x \omega_1 \\ \bar{t}_y \omega_2 \end{Bmatrix} dC \tag{9.16}$$

9.2.2 The Galerkin FEM

In the Galerkin FEM, one sets $\omega_1 = \omega_2 = N_j, \; j = 1 \ldots m$, leading to

$$\left[\int_{\Omega^{(e)}} \begin{bmatrix} \dfrac{\partial N_j}{\partial x} & 0 & \dfrac{\partial N_j}{\partial y} \\ 0 & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_j}{\partial x} \end{bmatrix} [\mathbf{C}][\mathbf{B}]\, dS \right] \{\mathbf{d}\} = \int_{\Omega^{(e)}} \begin{bmatrix} N_j & 0 \\ 0 & N_j \end{bmatrix} \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dS + \int_{\Gamma^{(e)}} \begin{bmatrix} N_j & 0 \\ 0 & N_j \end{bmatrix} \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} dC$$

$$j = 1 \ldots m \tag{9.17}$$

or

FEM Equations for 2D Elasticity:

$$[\mathbf{K}]\{\mathbf{d}\} = \mathbf{f}$$

$$\text{where} \quad [\mathbf{K}] = \int_{\Omega^{(e)}} [\mathbf{B}]^T [\mathbf{C}][\mathbf{B}] dS, \quad \mathbf{f} = \int_{\Omega^{(e)}} [\mathbf{N}]^T \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dS + \int_{\Gamma^{(e)}} [\mathbf{N}]^T \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} dC \tag{9.18}$$

The **f** here is a force per unit length (normal to the plane). One can re-formulate the problem and explicitly include the thickness, t say, so that

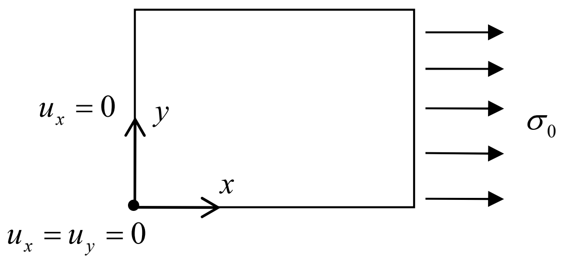
$$[\mathbf{K}] = t \int_{\Omega^{(e)}} [\mathbf{B}]^T [\mathbf{C}][\mathbf{B}] dS, \quad \mathbf{f} = t \int_{\Omega^{(e)}} [\mathbf{N}]^T \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} dS + t \int_{\Gamma^{(e)}} [\mathbf{N}]^T \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} dC$$

in which case **f** is a force.

9.3 Example Problems

Here, some example problems are considered, to illustrate the application of the above equations and the effectiveness of various elements.

9.3.1 Plate Extension



Consider a plate loaded along one edge by a constant stress σ_0. The plate is fixed along the opposite edge, but can slide vertically, as shown. Plane stress conditions are assumed. The exact solution to this problem is

$$[\boldsymbol{\sigma}] = \begin{Bmatrix} \sigma_0 \\ 0 \\ 0 \end{Bmatrix}, \qquad [\boldsymbol{\varepsilon}] = \begin{Bmatrix} \sigma_0 / E \\ -\nu\sigma_0 / E \\ 0 \end{Bmatrix}, \qquad \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \sigma_0 x / E \\ -\nu\sigma_0 y / E \end{bmatrix} \tag{9.19}$$

This is a simple linearly varying displacement field and constant stress and strain field.

Linear Triangular Elements

In terms of global coordinates, the displacement field within a linear triangular element is

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y \\ v &= \beta_1 + \beta_2 x + \beta_3 y \end{aligned} \tag{9.20}$$

The strains are then

$$\varepsilon_{xx} = \alpha_2, \quad \varepsilon_{yy} = \beta_3, \quad \gamma_{xy} = \alpha_3 + \beta_2 \tag{9.21}$$

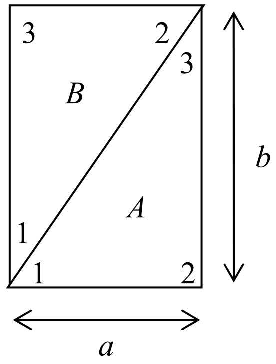
and for this reason the linear triangular element is also called the *constant strain triangle* (CST) element when used in the context of plane elasticity. The linear displacement field implies that the straight element edges remain straight after deformation. Further, because the strains are constant, the stresses are constant, so the equilibrium equations are satisfied exactly for this element in the absence of a body force. It can also be seen that compatibility is satisfied.

The exact solution to the plate extension problem involves a constant stress and strain field and so the CST element gives exact results.

The shape function $[\mathbf{N}]$ matrix and the strain-displacement $[\mathbf{B}]$ matrix are, in terms of local coordinates,

$$\{\mathbf{d}\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}, \quad [\mathbf{N}] = \begin{bmatrix} 1-\xi-\eta & 0 & \xi & 0 & \eta & 0 \\ 0 & 1-\xi-\eta & 0 & \xi & 0 & \eta \end{bmatrix} \tag{9.22}$$

$$[\mathbf{B}] = \frac{1}{2\Delta} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \tag{9.23}$$



The strain-displacement $[\mathbf{B}]$ matrix and the material property $[\mathbf{C}]$ matrix are independent of position, so, transforming the stiffness matrix integrals to local coordinates,

$$[\mathbf{K}] = [\mathbf{B}]^T[\mathbf{C}][\mathbf{B}]J\int_0^1\int_0^{1-\xi} d\eta d\xi = \Delta[\mathbf{B}]^T[\mathbf{C}][\mathbf{B}] \tag{9.24}$$

For a regular triangular mesh which includes the two types of element illustrated, A and B, the matrix multiplication leads to simple stiffness matrices:

<u>Element A</u>

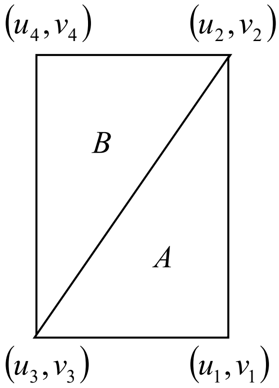
$$[\mathbf{K}^A] = \frac{E}{1-\nu^2}\frac{1}{2ab}\begin{bmatrix} b^2 & 0 & -b^2 & \nu ab & 0 & -\nu ab \\ & \frac{1}{2}(1-\nu)b^2 & \frac{1}{2}(1-\nu)ab & -\frac{1}{2}(1-\nu)b^2 & -\frac{1}{2}(1-\nu)ab & 0 \\ & & \frac{1}{2}(1-\nu)a^2+b^2 & -\frac{1}{2}(1+\nu)ab & -\frac{1}{2}(1-\nu)a^2 & \nu ab \\ & & & a^2+\frac{1}{2}(1-\nu)b^2 & \frac{1}{2}(1-\nu)ab & -a^2 \\ & & & & \frac{1}{2}(1-\nu)a^2 & 0 \\ & & & & & a^2 \end{bmatrix} \tag{9.25}$$

<u>Element B</u>

$$[\mathbf{K}^B] = \frac{E}{1-\nu^2}\frac{1}{2ab}\begin{bmatrix} \frac{1}{2}(1-\nu)a^2 & 0 & 0 & -\frac{1}{2}(1-\nu)ab & -\frac{1}{2}(1-\nu)a^2 & \frac{1}{2}(1-\nu)ab \\ & a^2 & -\nu ab & 0 & \nu ab & -a^2 \\ & & b^2 & 0 & -b^2 & \nu ab \\ & & & \frac{1}{2}(1-\nu)b^2 & \frac{1}{2}(1-\nu)ab & -\frac{1}{2}(1-\nu)b^2 \\ & & & & \frac{1}{2}(1-\nu)a^2+b^2 & -\frac{1}{2}(1+\nu)ab \\ & & & & & a^2+\frac{1}{2}(1-\nu)b^2 \end{bmatrix} \tag{9.26}$$

Let the complete plate be modeled with just the two elements illustrated, so that there are 8 degrees of freedom. Integrating along the edge $2-3$ of element A leads to the loads vector

$$\sigma_0 \int_{\Gamma^{(e)}} \begin{bmatrix} 1-\xi-\eta & 0 \\ 0 & 1-\xi-\eta \\ \xi & 0 \\ 0 & \xi \\ \eta & 0 \\ 0 & \eta \end{bmatrix} \begin{Bmatrix}1\\0\end{Bmatrix} dC = \sigma_0 \int_{\Gamma^{(e)}} \begin{bmatrix}1-\xi-\eta\\0\\\xi\\0\\\eta\\0\end{bmatrix} dC = b\sigma_0 \int_0^1 \begin{bmatrix}0\\0\\1-\eta\\0\\\eta\\0\end{bmatrix} d\eta = \frac{b\sigma_0}{2}\begin{bmatrix}0\\0\\1\\0\\1\\0\end{bmatrix} \tag{9.27}$$



which is equivalent to putting half the total force applied to the edge at each of the two edge-nodes.

Assembling the matrices and applying the essential boundary conditions $u_3 = v_3 = u_4 = 0$ leads to the global system

$$\begin{bmatrix} \tfrac{1}{2}(1-\nu)a^2 + b^2 & -\tfrac{1}{2}(1+\nu)ab & -\tfrac{1}{2}(1-\nu)a^2 & \nu ab & 0 \\ & a^2 + \tfrac{1}{2}(1-\nu)b^2 & \tfrac{1}{2}(1-\nu)ab & -a^2 & 0 \\ & & \tfrac{1}{2}(1-\nu)a^2 + b^2 & 0 & \nu ab \\ & \text{SYM} & & a^2 + \tfrac{1}{2}(1-\nu)b^2 & -\tfrac{1}{2}(1-\nu)b^2 \\ & & & & a^2 + \tfrac{1}{2}(1-\nu)b^2 \end{bmatrix}$$
$$\times \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ v_4 \end{bmatrix} = \frac{1-\nu^2}{E} ab^2 \sigma_0 \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} \tag{9.28}$$

leading to the exact solution

$$u_1 = u_2 = a\sigma_0 / E, \quad v_1 = 0, \quad v_2 = v_4 = -\nu b\sigma_0 / E \tag{9.29}$$

Within each element, the strains are obtained from (9.15), $\{\boldsymbol{\varepsilon}\} = [\mathbf{B}]\{\mathbf{d}\}$, and the stresses are obtained from the strains through (9.6), $\{\boldsymbol{\sigma}\} = [\mathbf{C}]\{\mathbf{e}\}$, leading to the exact solution (9.19).

Bilinear Quadrilateral (Q4) Elements

In terms of global coordinates, the displacement field within a bilinear quadrilateral element is

$$\begin{aligned} u &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy \\ v &= \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy \end{aligned} \tag{9.30}$$

and so the strains are

$$\varepsilon_{xx} = \alpha_2 + \alpha_4 y, \quad \varepsilon_{yy} = \beta_3 + \beta_4 x, \quad \gamma_{xy} = (\alpha_3 + \beta_2) + \alpha_4 x + \beta_4 y \tag{9.31}$$

The displacement field implies that the straight element edges do not remain straight after deformation, unless they are aligned with the global $x - y$ axes. Compatibility is satisfied but the equations of equilibrium are not necessarily satisfied exactly.

For the bilinear quadrilateral element, the displacement vector is

$$\{\mathbf{d}\} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T, \tag{9.32}$$

and the shape function $[\mathbf{N}]$ matrix in terms of local coordinates is

$$[\mathbf{N}] = \frac{1}{4}\begin{bmatrix} (1-\xi)(1-\eta) & 0 \\ 0 & (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) & 0 \\ 0 & (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) & 0 \\ 0 & (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) & 0 \\ 0 & (1-\xi)(1+\eta) \end{bmatrix}^T \tag{9.33}$$

The strain-displacement $[\mathbf{B}]$ matrix must be evaluated using the inverse Jacobian $\mathbf{J}^{-1}$:

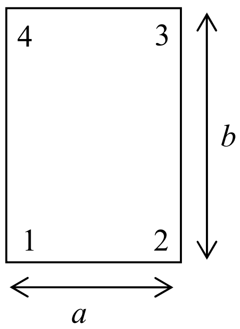
$$[\mathbf{B}] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \tag{9.34}$$

with

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} J_{11}^{-1} + \frac{\partial N_i}{\partial \eta} J_{12}^{-1}, \quad \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} J_{21}^{-1} + \frac{\partial N_i}{\partial \eta} J_{22}^{-1} \tag{9.35}$$

Taking a single rectangular element to model the plate, as illustrated below,

$$[\mathbf{B}] = \frac{1}{4ab} \begin{bmatrix} -b(1-\eta) & 0 & +b(1-\eta) & 0 & +b(1+\eta) & 0 & -b(1+\eta) & 0 \\ 0 & -a(1-\xi) & 0 & -a(1+\xi) & 0 & +a(1+\xi) & 0 & +a(1-\xi) \\ -a(1-\xi) & -b(1-\eta) & -a(1+\xi) & +b(1-\eta) & +a(1+\xi) & +b(1+\eta) & +a(1-\xi) & -b(1+\eta) \end{bmatrix} \tag{9.36}$$



The single element stiffness matrix is now

$$[\mathbf{K}] = ab \int_{-1}^{+1} \int_{-1}^{+1} [\mathbf{B}]^T [\mathbf{C}][\mathbf{B}] d\eta d\xi \tag{9.37}$$

Evaluating the integrals leads to the symmetric matrix

$$\frac{E}{1-\nu^2} \frac{1}{4ab} \begin{bmatrix} \frac{4}{3}\left(\alpha a^2 + b^2\right) & ab(+\nu+\alpha) & \frac{2}{3}\alpha a^2 - \frac{4}{3} b^2 & ab(+\nu-\alpha) & \\ & \frac{4}{3}\left(a^2 + \alpha b^2\right) & ab(-\nu+\alpha) & \frac{2}{3} a^2 - \frac{4}{3}\alpha b^2 & \\ & & \frac{4}{3}\left(\alpha a^2 + b^2\right) & ab(-\nu-\alpha) & \\ & & & \frac{4}{3}\left(a^2 + \alpha b^2\right) & \ldots \\ & & & & \end{bmatrix}$$

$$\cdots \left.\begin{matrix} -\frac{2}{3}\alpha a^2-\frac{2}{3}b^2 & ab(-\nu-\alpha) & -\frac{4}{3}\alpha a^2+\frac{2}{3}b^2 & ab(-\nu+\alpha) \\ ab(-\nu-\alpha) & -\frac{2}{3}a^2-\frac{2}{3}\alpha b^2 & ab(+\nu-\alpha) & -\frac{4}{3}a^2+\frac{2}{3}\alpha b^2 \\ -\frac{4}{3}\alpha a^2+\frac{2}{3}b^2 & ab(+\nu-\alpha) & -\frac{2}{3}\alpha a^2-\frac{2}{3}b^2 & ab(+\nu+\alpha) \\ ab(-\nu+\alpha) & -\frac{4}{3}a^2+\frac{2}{3}\alpha b^2 & ab(+\nu+\alpha) & -\frac{2}{3}a^2-\frac{2}{3}\alpha b^2 \\ \frac{4}{3}\left(\alpha a^2+b^2\right) & ab(+\nu+\alpha) & \frac{2}{3}\alpha a^2-\frac{4}{3}b^2 & ab(+\nu-\alpha) \\ & \frac{4}{3}\left(a^2+\alpha b^2\right) & ab(-\nu+\alpha) & \frac{2}{3}a^2-\frac{4}{3}\alpha b^2 \\ & & \frac{4}{3}\left(\alpha a^2+b^2\right) & ab(-\nu-\alpha) \\ & & & \frac{4}{3}\left(a^2+\alpha b^2\right) \end{matrix}\right] \tag{9.38}$$

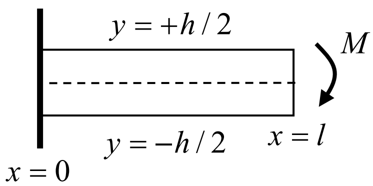
where $\alpha=(1-\nu)/2$. Applying the essential boundary conditions $u_1=v_2=u_4=0$ and using the loads vector

$$\{\mathbf{f}\}=\frac{b\sigma_0}{2}\{0 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0\}^T, \tag{9.39}$$

then leads to the exact solution.

9.3.2 Beam Bending

Consider next a beam built in at one end and subjected to a pure moment M at the other end. The exact solution to this problem for conditions of plane stress is



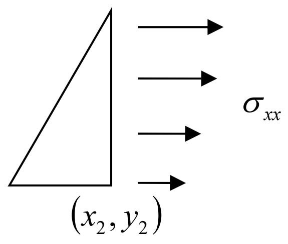
$$[\boldsymbol{\sigma}]=\frac{12M}{h^3}\begin{Bmatrix} y \\ 0 \\ 0 \end{Bmatrix}, \qquad [\boldsymbol{\varepsilon}]=\frac{12M}{h^3E}\begin{Bmatrix} y \\ -\nu y \\ 0 \end{Bmatrix} \tag{9.40}$$

It is not possible to get an exact solution for the displacements; an approximate solution is

$$u=\frac{12M}{Eh^3}[xy], \qquad v=-\frac{12M}{Eh^3}\left[\frac{1}{2}\left(x^2+\nu y^2\right)\right] \tag{9.41}$$

which when differentiated give the exact strains; they satisfy the condition $u(0,0)=v(0,0)=0$ but do not satisfy $v|_{x=0}=0$ unless $y=0$.

For the right-angled CST element shown below, the load vector is {▲Problem 1}



$$\mathbf{f} = \frac{2Mb}{h^3}\begin{bmatrix} 0 & 0 & 3y_2 + b & 0 & 3y_2 + 2b & 0 \end{bmatrix}^T \qquad (9.42)$$

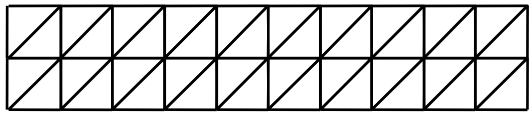
where y_2 is the y coordinate of the bottom right vertex, as illustrated. For a single element, with $y_2 = -h/2$, $b = h$, this gives rise to forces of $\pm M/h$ at the two end-nodes. The loads vector for the Q4 element is the same (with two extra zero entries at the bottom of the vector).

Clearly, from (9.21), the CST element does not give the exact solution, and indeed does not perform well. From (9.31), the Q4 element cannot model the linear variation of the ε_{yy} strain in y. Further, since α_4 in (9.31) needs to be non-zero to model the linear variation of ε_{xx} in y, it develops a non-zero shear strain γ_{xy}. Thus, not only is the bending resisted by the flexural stresses σ_{xx}, it is resisted by shear stresses σ_{xy} and the consequence is that the Q4 element is overly stiff in bending.

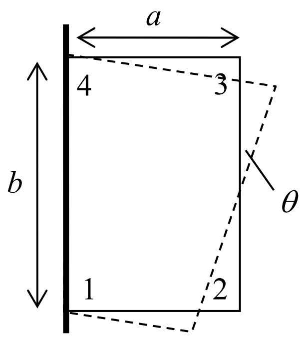
Results for the CST and Q4 elements for the displacement at the lower right hand end of the beam are given in the following table. The geometry is $l = 10$, $h = 2$. In this table, n is the number of nodes, N is the number of elements and dof is the number of degrees of freedom (before application of the essential BC's). Results are given for various $m_x \times m_y$ meshes, these being the number of divisions of the beam in the x and y directions respectively; for example, the 10×2 mesh of CST elements is shown below. The Poisson's ratio was taken to be $\nu = 0.3$.

			CST			Q4		
Mesh	n	dof	N	$\frac{E}{M} u_x \big\vert_{(l,-h/2)}$	$\frac{E}{M} u_y \big\vert_{(l,-h/2)}$	N	$\frac{E}{M} u_x \big\vert_{(l,-h/2)}$	$\frac{E}{M} u_y \big\vert_{(l,-h/2)}$
5×1	12	24	10	-3.55117	-17.19328	5	-10.11111	-50.55555
10×2	33	66	40	-8.18828	-40.45249	20	-13.35359	-66.73442
20×2	63	126	80	-9.18705	-45.68630	40	-14.28070	-71.40650
50×2	153	306	200	-9.47978	-47.33016	100	-14.56502	-72.84635
50×10	561	1122	1000	-14.46915	-72.45387	500	-14.90095	-74.60057
100×20	2121	4242	4000	-14.84735	-74.33781	2000	-14.96151	-74.89682
Approx. Solution (Eqn. 9.36)				**-15.00000**	**-75.22500**		**-15.00000**	**-75.22500**

Table: Performance of CST and Q4 elements for Pure Bending



Locking



If the aspect ratio of the Q4 element is allowed to get very large, its stiffness becomes very inaccurate. For example, consider the beam bending modelled by a single Q4 element. The FE solution for the displacements is then

$$u_2 = -u_3 = -\frac{6(1-\nu^2)Ma}{E(\alpha a^2 + b^2)}, \quad v_2 = v_3 = au_2 \qquad (9.43)$$

The element deforms as indicated by the dotted lines. The true solution for the angle θ indicated is, from the simple beam theory, $12\overline{M}a / b^3 E$, where $\overline{M}$ is the true moment required. Setting this θ equal to that obtained from the FE solution gives

$$M = \frac{1}{1+\nu}\left[\frac{1}{2}\left(\frac{a}{b}\right)^2 + \frac{1}{1-\nu}\right]\overline{M} \qquad (9.44)$$

Thus for large aspect ratio a/b, the moment required in the FE model tends to infinity and the model becomes infinitely stiff, a phenomenon known as *locking*.

The Improved Bilinear Quadrilateral (Q6) Element

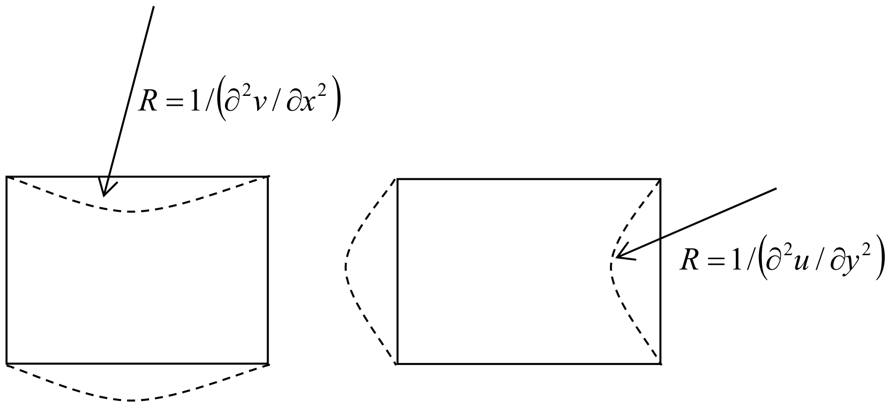
The Q6 element was introduced in order to overcome the problems associated with the Q4 element in bending. It has the same four nodes as the Q4 element, but six degrees of freedom. The displacements within the element are given by

$$\begin{aligned} u &= \sum_{i=1}^{4} N_i u_i + (1-\xi^2)\overline{u}_5 + (1-\eta^2)\overline{u}_6 \\ v &= \sum_{i=1}^{4} N_i v_i + (1-\xi^2)\overline{v}_5 + (1-\eta^2)\overline{v}_6 \end{aligned} \qquad (9.45)$$

where the N_i are the same as the Q4 shape functions. The $\bar{u}_5, \bar{u}_6$, $\bar{v}_5, \bar{v}_6$, are new degrees of freedom termed *internal* degrees of freedom – they are not nodal displacements. This new interpolation allows straight edges to deform into curves; in fact, considering a rectangular element with edges aligned with the global axes,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{8}{b^2}\bar{u}_6, \qquad \frac{\partial^2 v}{\partial x^2} = -\frac{8}{a^2}\bar{v}_5 \tag{9.46}$$

These are constant curvatures within an element, the inverses of the radii of curvature shown in the figure below.

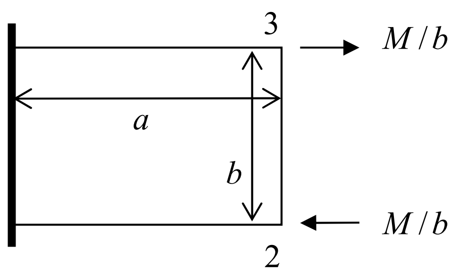


If this element is rectangular with sides aligned with the global axes, then a state of pure bending can be represented exactly. The loads vector for the single element shown is

$$\mathbf{f} = \frac{2Mb}{h^3}\begin{bmatrix}0 & 0 & 3y_2+b & 0 & 3y_2+2b & 0 & 0 & 0 & 0 & 0 & 4y_2+2b & 0\end{bmatrix}^T \tag{9.47}$$

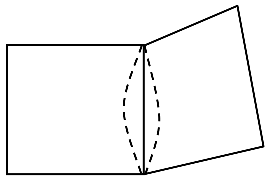
and the corresponding FE solution is

$$u_2 = -u_3 = -\frac{6Ma}{Eb^2}, \quad v_2 = v_3 = -\frac{6Ma^2}{Eb^3}, \quad \bar{u}_5 = \bar{u}_6 = 0, \quad \bar{v}_5 = \frac{3Ma^2}{2Eb^3}, \quad \bar{v}_6 = \frac{3M\nu}{2Eb} \tag{9.48}$$



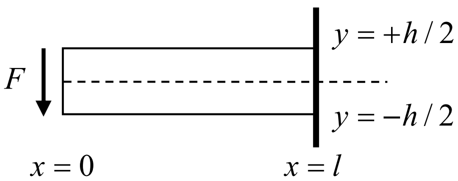
which leads to the exact expressions for the strains within the element, and the exact expression for the radius of curvature, $R = -h^3 E/12M$.

Although the compatibility equation (9.5) is satisfied by (9.45), the Q6 element is still incompatible in the sense that the edges of two adjacent elements which are initially coincident can deform to two separate curves, corresponding to an overlap of material, as illustrated here. This problem dissipates as the mesh is refined and a state of constant strain is approached within each element.



9.3.3 The Cantilever Beam

Consider now a more challenging problem to model, that of a cantilever beam subjected to a shear force F at its end. The exact solution to this problem for conditions of plane stress is



$$[\boldsymbol{\sigma}] = \frac{6F}{h^3}\begin{Bmatrix} 2xy \\ 0 \\ (h/2)^2 - y^2 \end{Bmatrix}, \tag{9.49}$$

$$[\boldsymbol{\varepsilon}] = \frac{6F}{h^3 E}\begin{Bmatrix} 2xy \\ -2\nu xy \\ 2(1+\nu)\left((h/2)^2 - y^2\right) \end{Bmatrix} \tag{9.50}$$

It is not possible to get an exact expression for the displacements; they are given by

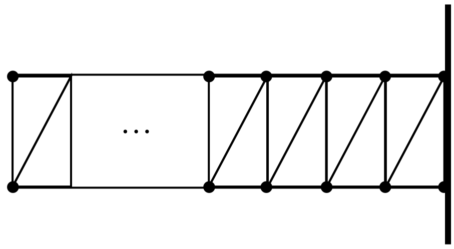
$$\begin{aligned} u &= \frac{2F}{Eh^3}\left[-(2+\nu)y^3 - 3\left(l^2 - x^2\right)y + \tfrac{3}{2}(1+\nu)h^2 y\right] + \overline{A}y \\ v &= \frac{2F}{Eh^3}\left[-x^3 - 3\nu xy^2 + 3l^2 x - 2l^3\right] + \overline{A}(l - x) \end{aligned} \tag{9.51}$$

where $\overline{A}$ is constant. These expressions satisfy the strain-displacement relations and satisfy $u(l,0) = v(l,0) = 0$. It is not possible to ensure (9.51) satisfy $u = v = 0$ all along the right hand end; $\overline{A}$ can be determined by enforcing some condition there, for example zero slope $\partial v/\partial x|_{x=l} = 0$ gives $\overline{A} = 0$, whereas zero slope $\partial u/\partial y|_{x=l} = 0$ gives $\overline{A} = -3F(1+\nu)/Eh$. The exact solution is somewhere between these extremes. The displacements at the lower left hand corner are

$$u\big|_{(0,-h/2)} = \frac{F}{E}\left[+3(l/h)^2\right] + O(1), \qquad v\big|_{(0,-h/2)} = \frac{F}{E}\left[-4(l/h)^3\right] + O(l/h) \qquad (9.52)$$

so that, if the beam is slender, $l/h >> 1$, and the first terms here will dominate.

The exact solution involves a stress and strain field which vary through the thickness of the beam. As with pure bending, a single CST element in the thickness direction is not able to capture this variation and in fact a model such as that shown here returns very poor results. No matter how many CST elements are used in the horizontal direction, the solution is very inaccurate, for example with $l/h = 10$ and 200 elements, the error in $u\big|_{(0,-h/2)}$ is about 70%. Even when the number of elements in the thickness direction is increased, the results are not very good, unless many hundreds of elements are used. The results using the Q4 element are better. Results for both CST and Q4 elements for the displacement at the lower left hand end are given in the following table. Again, $l = 10$, $h = 2$ and $\nu = 0.3$.



			CST			Q4		
Mesh	n	dof	N	$\frac{E}{F}u_x\big\|_{(0,-h/2)}$	$\frac{E}{F}u_y\big\|_{(0,-h/2)}$	N	$\frac{E}{F}u_x\big\|_{(0,-h/2)}$	$\frac{E}{F}u_y\big\|_{(0,-h/2)}$
5×1	12	24	10	16.32264	-125.0169	5	50.55555	-346.6667
10×2	33	66	40	39.75799	-279.4301	20	66.63406	-455.2244
20×2	63	126	80	45.20658	-315.1775	40	71.29923	-487.3444
50×2	153	306	200	47.07871	-327.0089	100	72.73698	-497.2866
50×10	561	1122	1000	72.53138	-496.2589	500	74.80125	-511.0163
100×20	2121	4242	4000	74.53338	-509.3301	2000	75.15152	-513.2310
Approx. Solution (Eqn. 9.52)				**75.00000**	**-500.0000**		**75.00000**	**-500.0000**

Table: Performance of CST and Q4 elements for Tip Loading of a Cantilever

As expected, the Q6 element gives better results tan either the CST or Q4.

9.4 Problems

1. Derive the loads vector **f** for the CST triangle in the beam bending problem.