4 Finite Element Methods for Partial Differential Equations

Ordinary Differential Equations (ODEs) have been considered in the previous two Chapters. Here, Partial Differential Equations (PDEs) are examined. Taking x and t to be the independent variables, a general second-order PDE is

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial t} + c\frac{\partial^2 u}{\partial t^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial t} + fu = g \tag{4.1}$$

PDEs are classified according to the value of $b^2 - ac$:

$$b^2 - ac = \begin{cases} > 0 & \text{hyperbolic} \\ = 0 & \text{parabolic} \\ < 0 & \text{elliptic} \end{cases} \tag{4.2}$$

Two special cases of the PDE (4.1) will be examined here, the most commonly encountered ones in applications; these are the first order (in t) parabolic system

$$a\frac{\partial^2 u}{\partial x^2} + d\frac{\partial u}{\partial t} + fu = g \tag{4.3}$$

and the second order (in t) hyperbolic system

$$a\frac{\partial^2 u}{\partial x^2} + c\frac{\partial^2 u}{\partial t^2} + fu = g, \quad ac < 0 \tag{4.4}$$

In applications, x will usually represent a spatial coordinate and t will represent time. This terminology is used below for these variables.

The Galerkin Finite Element Method is used to reduce these PDEs to a system of ODEs, which can then be solved using standard ODE solver algorithms.

4.1 First Order Systems

Here, the first order parabolic equation (4.3) is discussed. In particular, consider the following problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{B.C.} \quad \begin{matrix} u(0,t) = 0 \\ u(1,t) = 1 \end{matrix}, \quad \text{I.C.} \quad u(x,0) = \sin \pi x + x \tag{4.5}$$

[exact solution: $u(x,t) = \sin \pi x \exp(-\pi^2 t) + x$]

4.1.1 FE equations for First Order Systems

The weighted residual form of (4.5) is

$$\int_0^1 \frac{\partial u}{\partial t} w dx + \int_0^1 \frac{\partial u}{\partial x} \frac{\partial w}{\partial x} dx = \left[\frac{\partial u}{\partial x} w \right]_0^1 \tag{4.6}$$

The Galerkin procedure and shape functions are used to discretise the *space* variable in (4.6) only; the nodal values are functions of t. For a linear element, Fig. 4.1, let

$$u(x,t) = u_i(t) N_1(x) + u_{i+1}(t) N_2(x)$$

$$\frac{\partial u(x,t)}{\partial x} = u_i(t) \frac{\partial N_1(x)}{\partial x} + u_{i+1}(t) \frac{\partial N_2(x)}{\partial x} \tag{4.7}$$

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial u_i(t)}{\partial t} N_1(x) + \frac{\partial u_{i+1}(t)}{\partial t} N_2(x)$$

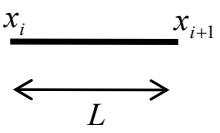

Figure 4.1: A Linear Element

These lead to two equations, one for each weight N_j,

$$\frac{\partial u_i}{\partial t}\int_{x_i}^{x_{i+1}} N_1 N_j dx + \frac{\partial u_{i+1}}{\partial t}\int_{x_i}^{x_{i+1}} N_2 N_j dx + u_i \int_{x_i}^{x_{i+1}} \frac{\partial N_1}{\partial x}\frac{\partial N_j}{\partial x} dx + u_{i+1}\int_{x_i}^{x_{i+1}} \frac{\partial N_2}{\partial x}\frac{\partial N_j}{\partial x} dx = \left[\frac{\partial u}{\partial x} N_j\right]_{x_i}^{x_{i+1}} \qquad j = 1,2 \tag{4.8}$$

Evaluating the integrals leads to the element equations

$$\frac{L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{u}_i \\ \dot{u}_{i+1} \end{bmatrix} + \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} = \begin{bmatrix} -\partial u/\partial x(x_i) \\ +\partial u/\partial x(x_{i+1}) \end{bmatrix} \tag{4.9}$$

element capacitance matrix

Element stiffness matrix

The difference between the FE equations for a first order system and those for the standard linear (ODE) system, is the appearance of the capacitance matrix[1] **C**.

After assembly, one has the system of *ordinary* differential equations

$$\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \tag{4.10}$$

These equations can be solved in a number of different ways (see below).

4 linear elements

Assembling the global 5×5 matrices for the case of four linear elements, applying the boundary conditions $u(0) = u_1 = 0,\ u(1) = u_5 = 1$, noting that $\dot{u}_1 = 0, \dot{u}_5 = 0$, and eliminating the first and last equations leads to {▲Problem 1}

$$\frac{1}{24}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}\begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{bmatrix} + 4\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \tag{4.11}$$

[1] so-called because this first order system arises in heat conduction problems, and this **C** matrix involves the specific heat capacity of materials

which is a system of three coupled first order ODEs, which can be solved subject to the initial conditions $u(x,0) = \sin \pi x + x$.

Example

As another example, consider the differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\alpha \frac{\partial u}{\partial x} \right) = f(x) \tag{4.12}$$

which is a general form of a first order in time equation which arises in many important problems, including transient heat conduction, diffusion, flow through channels and other applications.

Forming the weighted residual integral and using linear shape functions, $\tilde{u}(x,t) = N_1(x)u_1(t) + N_2(x)u_2(t)$,

$$\begin{aligned}
&\frac{du_1}{dt}\int_0^l N_1 N_1 dx + \frac{du_2}{dt}\int_0^l N_2 N_1 dx \\
&\quad + u_1 \int_0^l \alpha \frac{dN_1}{dx}\frac{dN_1}{dx}dx + u_2 \int_0^l \alpha \frac{dN_2}{dx}\frac{dN_1}{dx}dx = \left[\alpha \frac{\partial u}{\partial x} N_1 \right]_0^l + \int_0^l f\, N_1 dx \\
&\frac{du_1}{dt}\int_0^l N_1 N_2 dx + \frac{du_2}{dt}\int_0^l N_2 N_2 dx \\
&\quad + u_1 \int_0^l \alpha \frac{dN_1}{dx}\frac{dN_2}{dx}dx + u_2 \int_0^l \alpha \frac{dN_2}{dx}\frac{dN_2}{dx}dx = \left[\alpha \frac{\partial u}{\partial x} N_2 \right]_0^l + \int_0^l f\, N_2 dx
\end{aligned} \tag{4.13}$$

Transforming to local coordinates, from $x = [x_i, x_{i+1}]$ to $\xi = [-1, +1]$, and also approximating the "loading" function $f(x)$ by a linear interpolation, $f(x) = f_1 N_1(x) + f_2 N_2(x)$, and taking α to eb a constant for the sake of illustration,

$$\begin{aligned}
&\dot{u}_1 \left[\frac{L}{2}\int_{-1}^{+1} N_1 N_j d\xi \right] + \dot{u}_2 \left[\frac{L}{2}\int_{-1}^{+1} N_2 N_j d\xi \right] \\
&\quad + u_1 \alpha \left[\frac{2}{L}\int_{-1}^{+1} \frac{dN_1}{d\xi}\frac{dN_j}{d\xi}d\xi \right] + u_2 \alpha \left[\frac{2}{L}\int_{-1}^{+1} \frac{dN_2}{d\xi}\frac{dN_j}{d\xi}d\xi \right] \qquad , \qquad j = 1,2 \\
&\quad = \left[\alpha \frac{\partial u}{\partial x} N_j \right]_{-1}^{+1} + f_1 \left[\frac{L}{2}\int_{-1}^{+1} N_1\, N_j d\xi \right] + f_2 \left[\frac{L}{2}\int_{-1}^{+1} N_2\, N_j d\xi \right]
\end{aligned} \tag{4.14}$$

Evaluating all the integrals using the results of the Appendix to Chapter 2, section 2.12.1, leads to the element equations

$$\frac{L}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\end{bmatrix}+\alpha\frac{1}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}=\alpha\begin{bmatrix}-u'(-1)\\ +u'(+1)\end{bmatrix}+\frac{L}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}f_1\\ f_2\end{bmatrix} \tag{4.15}$$

The final global system of equations is then

$$\begin{aligned}\frac{L}{6}\begin{bmatrix}2 & 1 & 0 & \cdots & 0 & 0\\ 1 & 4 & 1 & \cdots & 0 & 0\\ 0 & 1 & 4 & \cdots & 0 & 0\\ \vdots & & & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 2\end{bmatrix}\begin{bmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{u}_3\\ \vdots\\ \dot{u}_{n+1}\end{bmatrix}+\frac{\alpha}{L}\begin{bmatrix}1 & -1 & 0 & \cdots & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \vdots & & & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ \vdots\\ u_{n+1}\end{bmatrix}\\ =\alpha\begin{bmatrix}-u'(-1)\\ 0\\ 0\\ \vdots\\ +u'(+1)\end{bmatrix}+\frac{L}{6}\begin{bmatrix}2 & 1 & 0 & \cdots & 0 & 0\\ 1 & 4 & 1 & \cdots & 0 & 0\\ 0 & 1 & 4 & \cdots & 0 & 0\\ \vdots & & & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 2\end{bmatrix}\begin{bmatrix}f_1\\ f_2\\ f_3\\ \vdots\\ f_{n+1}\end{bmatrix}\end{aligned} \tag{4.16}$$

Let us assume that the boundary conditions are

$$u_1=\overline{u}_1,\quad \left.\frac{\partial u}{\partial x}\right|_{x_{n+1}}=\overline{u}'_{n+1} \tag{4.17}$$

The natural boundary condition can be applied by directly replacing the term $u'(+1)$ in the right-hand side vector. The essential boundary condition can be applied by replacing the first row as follows:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{L}{6} & 4\frac{L}{6} & \frac{L}{6} & \cdots & 0 & 0 \\ 0 & \frac{L}{6} & 4\frac{L}{6} & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{L}{6} & 2\frac{L}{6} \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \vdots \\ \dot{u}_{n+1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{\alpha}{L} & 2\frac{\alpha}{L} & -\frac{\alpha}{L} & \cdots & 0 & 0 \\ 0 & -\frac{\alpha}{L} & 2\frac{\alpha}{L} & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\alpha}{L} & \frac{\alpha}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n+1} \end{bmatrix}$$
$$= \begin{bmatrix} \overline{u}_1 \\ 0 \\ 0 \\ \vdots \\ +\alpha\overline{u}'_{n+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{L}{6} & 4\frac{L}{6} & \frac{L}{6} & \cdots & 0 & 0 \\ 0 & \frac{L}{6} & 4\frac{L}{6} & \cdots & 0 & 0 \\ \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{L}{6} & \frac{L}{3} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n+1} \end{bmatrix} \tag{4.18}$$

Note that we have retained a "1" in the C_{11} element of the capacitance matrix **C**, otherwise it will be singular. The first row states that $\dot{u}_1 + u_1 = \overline{u}_1$. The general solution to this differential equation is $u_1 = Ae^{-t} + \overline{u}_1$; with the initial condition that $u_1(0) = \overline{u}_1$, it in effect states that $u_1 = \overline{u}_1$ for all time.

4.1.2 Eigenvalues and Eigenvectors

Before going on to discuss the various possible ways of solving the system 4.11, 4.18, in §4.1.3 below, it is worthwhile discussing the associated eignevalue problem. An eigenvalue analysis of the PDE (4.5) *involves the boundary conditions but disregards the initial conditions*. Although the initial conditions are not considered, and so the full solution is not obtained, nevertheless the analysis can furnish much useful information. For example, eigenvalues and eigenvectors often have a real physical significance for the problem at hand, and eigenvalues are related to the stability of numerical solution procedures for the associated FE equations (see below).

Consider first a single degree of freedom of (4.10): $c\dot{u} + ku = f$, which has the solution $u(t) = f/k + e^{-\lambda t}$, where $\lambda = k/c$. The transient solution decays and after a sufficient amount of time the solution approaches the steady-state solution $u_s = f/k$. A solution to the complete system can be obtained by assuming a similar behaviour:

$$\mathbf{u}(t) = \mathbf{u}_s + \overline{\mathbf{u}}e^{-\lambda t} \tag{4.19}$$

Here, $\overline{\mathbf{u}} = \left[\overline{u}_1 \;\; \overline{u}_2 \;\; \cdots \;\; \overline{u}_n\right]^{\mathrm{T}}$ is called an *eigenvector* and λ an *eigenvalue*. Substituting into the system of equations (4.10) gives

$$\left[\mathbf{K} - \lambda\mathbf{C}\right]\overline{\mathbf{u}} = 0 \tag{4.20}$$

This is a system of $n \times n$ equations in the n nodal values of $\overline{\mathbf{u}}$. From Linear Algebra, such a system of homogeneous equations only has a (non-zero) solution if the determinant of the coefficient matrix is zero, that is

$$\left|\mathbf{K} - \lambda\mathbf{C}\right| = 0 \tag{4.21}$$

Eqn. 4.21 is a polynomial of the nth order and so has n solutions for the eigenvalue[2] λ; there is one eigenvalue for each degree of freedom of the system. Corresponding to each of the n eigenvalues $\lambda^{(j)}$ there is an eigenvector $\overline{\mathbf{u}}^{(j)}$. Each pair $\lambda^{(j)}, \overline{\mathbf{u}}^{(j)}$, corresponds to a certain *mode* of the system. The complete solution is a linear combination of these modes:

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{u}_s + \sum_{j=1}^{n} \beta_j \overline{\mathbf{u}}^{(j)} e^{-\lambda^{(j)} t} \\ u_i(t) &= u_{is} + \sum_{j=1}^{n} \beta_j \overline{u}_i^{(j)} e^{-\lambda^{(j)} t} \end{aligned} \tag{4.22}$$

for the nodes $i = 1, 2, \cdots, n$; the coefficients β_j depend on the initial conditions.

Mesh Size

In first-order linear problems, it can be shown that $\lambda_{\max} = \mathrm{O}\left(1/h^2\right)$, where h is a mesh length parameter (for example element-length). For example, consider the FE equations for a single linear element, Eqns (4.9). Then

$$\left|\mathbf{K} - \lambda\mathbf{C}\right| = \begin{vmatrix} \dfrac{1}{L} - \lambda\dfrac{L}{3} & -\dfrac{1}{L} - \lambda\dfrac{L}{6} \\ -\dfrac{1}{L} - \lambda\dfrac{L}{6} & \dfrac{1}{L} - \lambda\dfrac{L}{3} \end{vmatrix} = -\lambda + \frac{L^2}{12}\lambda^2 = 0 \quad \rightarrow \quad \lambda = 0, \frac{12}{L^2} \tag{4.23}$$

[2] it can be proved that these eigenvalues are also the eigenvalues of the matrix $\mathbf{C}^{-1}\mathbf{K}$

It can be seen that $\lambda_{\max} \propto 1/L^2$ so that, as the mesh gets very fine, the maximum eigenvalue gets very large. The consequences of this fact will be discussed further below.

4.1.3 Direct Integration for First Order Systems

A number of different direct integration methods are available for the integration of the first order system (4.10), for example,

1. Explicit Euler's method
2. Implicit Euler's method
3. Semi-implicit Euler's method
4. Predictor-Corrector method
5. Methods based on Runga-Kutta formulae

1. Explicit Euler's method

To derive the explicit Euler's method, first expand $u_i(t)$ in a Taylor series, *i* referrring to a particular node:

$$u_i(t+\Delta t) = u_i(t) + \Delta t \dot{u}_i(t) + \frac{1}{2}(\Delta t)^2 \ddot{u}_i(t) + \cdots \tag{4.24}$$

The time derivative at time *t* can then be approximated by the forward difference approximation[3]

$$\dot{u}_t = \frac{u_{t+\Delta t} - u_t}{\Delta t} \tag{4.25}$$

In (4.25), terms of order $\mathrm{O}(\Delta t)$ have been neglected from (4.22), that is, the *truncation error* is proportional to Δt. The FE equations are now written at time *t*, $\mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t)$, and then rearranged as

[3] note that there are many other explicit formulae, each derived from different finite difference Taylor expansion formulae; some of these are discussed further on

$$\mathbf{C}\mathbf{u}(t+\Delta t) = \mathbf{C}\mathbf{u}(t) + \Delta t\left[\mathbf{F}(t) - \mathbf{K}\mathbf{u}(t)\right] \tag{4.26}$$

For the purpose of coding, the equation can be rewritten using $\mathbf{u}(t+\Delta t) = \mathbf{u}(t) + \Delta\mathbf{u}$:

Explicit Euler Algorithm:

$$\begin{aligned} &\mathbf{C}\Delta\mathbf{u} = \mathbf{R}(t) \\ &\quad \text{where } \mathbf{R}(t) = \Delta t\left[\mathbf{F}(t) - \mathbf{K}\mathbf{u}(t)\right] \\ &\mathbf{u}(t+\Delta t) = \mathbf{u}(t) + \Delta\mathbf{u} \end{aligned} \tag{4.27}$$

The right-hand side here is known: Δt is chosen by the user, **K** is constant for all time, **F** is a known "loading" term which is specified, and **u** is known at time *t*. The left-hand side **C** is also a constant for all time. The algorithm is started by specifying initial conditions at all the nodes: $\mathbf{u}(0)$.

Note that the cost of the integration, that is, the number of operations required, is directly proportional to the number of time steps required for solution. It follows that the selection of an appropriate time step in direct integration is of much importance.

Considering a one-dimensional case for illustrative purposes, consider the ODE

$$\frac{du}{dt} + \lambda u = f, \quad u(0) = \overline{u}_0 \tag{4.28}$$

Substituting $\dot{u}_t = \left(u_{t+\Delta t} - u_t\right)/\Delta t$ into Eqn. 4.28 gives

$$u(t+\Delta t) = (1-\lambda\Delta t)u(t) + f\Delta t \tag{4.29}$$

which leads to, summing the geometric series,

$$\begin{aligned} u(n\Delta t) &= (1-\lambda\Delta t)^n u(0) + f\Delta t\sum_{i=0}^{n-1}(1-\lambda\Delta t)^i \\ &= (1-\lambda\Delta t)^n u(0) + \frac{f}{\lambda}\left[1-(1-\lambda\Delta t)^n\right] \end{aligned} \tag{4.30}$$

This is plotted in Fig. 4.2 for $\lambda = 2$, $f = 1$, $\overline{u}_0 = 1$; for $\Delta t = 1/3$ and $\Delta t = 1.05$, together with the exact solution $u(t) = \frac{1}{2}\left(1+e^{-2t}\right)$.

The solution is fairly accurate for $\Delta t = 1/3$ (the solution is very close to the exact solution for $\Delta t < 0.1$). On the other hand, when the time step is as large as $\Delta t = 1.05$, the solution is highly inaccurate; this issue is explained further below.

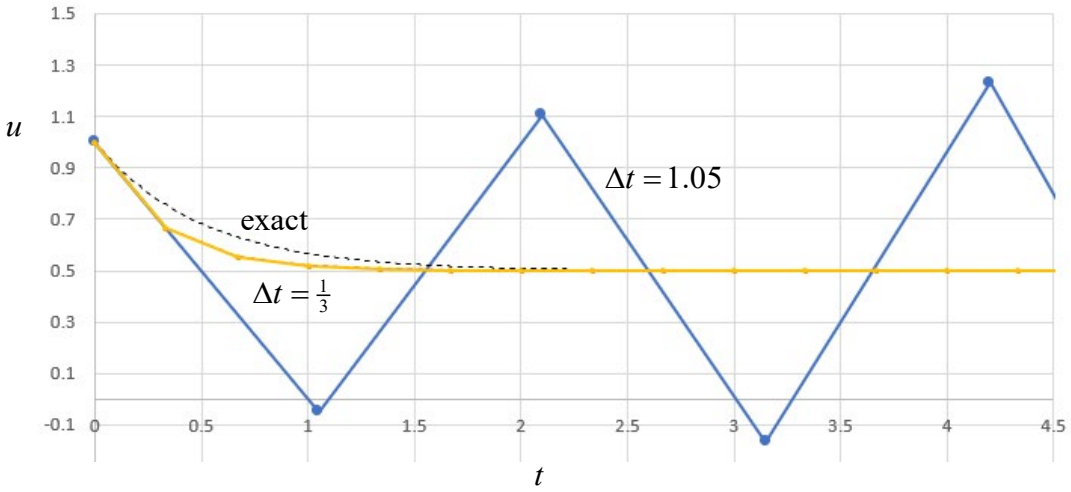

Figure 4.2: Explicit Euler scheme for the solution of an ODE

Matrix Lumping

The inversion of the explicit Euler equations can be greatly speeded up by having **C** diagonal. Altering **C** so that it is diagonal is called **matrix lumping**. There is no one generally accepted method, or theory, of matrix lumping – rather it is an *ad hoc* procedure, which happens not to introduce too significant an error. As an example, considering the earlier example, Eqns (4.11), the usual way to lump the global **C** matrix is as follows:

$$\frac{1}{24}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \rightarrow \frac{1}{24}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \tag{4.31}$$

The matrix on the left is called the **consistent matrix**, that on the right the **lumped matrix**.

Stability

The FE equations derived above are exact in the sense that if they are solved exactly, they will give the correct (approximate, FE) solution. However, the equations must be solved numerically, using for example the explicit Euler approximation of the time derivative, Eqn. 4.25. This and other approximations will lead to numerical errors (along with the

inevitable and rounding errors) in the various terms and equations. The question then arises: if there are some numerical/rounding errors in our calculations, will we still get (approximately) the correct solution?

A solution algorithm which is **stable** is one which remains close to the correct solution, i.e. errors in the result at one time step are damped down into the following time steps. An **unstable** algorithm, on the other hand, is one where errors at one time step are magnified in subsequent time-steps, causing the solution to diverge catastrophically.

Examining the one-dimensional case, Eqn. 4.28, $\dot{u} + \lambda u = f$, the general solution is of exponential form. In the case of constant *f*, the solution is $u(t) = \left[u(0) - f/\lambda\right]e^{-\lambda t} + f/\lambda$. We will assume at the outset that $\lambda > 0$; otherwise the solution will grow exponentially and we will usually be interested in practical problems involving physical systems which decay. In that case, the exact solution decays towards the steady-state $u_s = f/\lambda$. The explicit Euler numerical approximation of this exact solution is, on the other hand, given by Eqn. 4.30, $u(n\Delta t) = (1-\lambda\Delta t)^n u(0) + \frac{f}{\lambda}\left[1-(1-\lambda\Delta t)^n\right]$. It can be seen that the term $(1-\lambda\Delta t)^n$ is critical in the sense that, if $|1-\lambda\Delta t| > 1$, this term will grow without bound with successive time steps. Thus it appears that one requires that $|1-\lambda\Delta t| < 1$ for the solution to decay as required, i.e. $-1 < 1-\lambda\Delta t < 1$. With $\lambda > 0$, this implies that we must have

$$\Delta t < \frac{2}{\lambda} \tag{4.32}$$

for the solution to decay "correctly". It is for this reason that the solution diverged in Fig. 4.2 for the case of $\Delta t = 1.05 > 2/\lambda$ with $\lambda = 2$.

To examine the stability of algorithms associated with the general first order in time partial differential equation 4.12, here are listed the FE equations for various internal nodes in the mesh resulting from the use of linear elements (see Eqns. 4.16), neglecting the forcing vector $f(x)$, which does not affect the stability:

$$
\begin{aligned}
\frac{L}{6}\left\{\dot{u}_{i-3}+4\dot{u}_{i-2}+\dot{u}_{i-1}\right\}+\frac{\alpha}{L}\left\{-u_{i-3}+2u_{i-2}-u_{i-1}\right\}=0\\
\frac{L}{6}\left\{\dot{u}_{i-2}+4\dot{u}_{i-1}+\dot{u}_{i}\right\}+\frac{\alpha}{L}\left\{-u_{i-2}+2u_{i-1}-u_{i}\right\}=0\\
\frac{L}{6}\left\{\dot{u}_{i-1}+4\dot{u}_{i}+\dot{u}_{i+1}\right\}+\frac{\alpha}{L}\left\{-u_{i-1}+2u_{i}-u_{i+1}\right\}=0\\
\frac{L}{6}\left\{\dot{u}_{i}+4\dot{u}_{i+1}+\dot{u}_{i+2}\right\}+\frac{\alpha}{L}\left\{-u_{i}+2u_{i+1}-u_{i+2}\right\}=0\\
\frac{L}{6}\left\{\dot{u}_{i+1}+4\dot{u}_{i+2}+\dot{u}_{i+3}\right\}+\frac{\alpha}{L}\left\{-u_{i+1}+2u_{i+2}-u_{i+3}\right\}=0
\end{aligned}
\tag{4.33}
$$

Examining the lumped capacitance matrix, and the explicit Euler representation 4.25:

$$
\begin{aligned}
u_{i-2}\left(t+\Delta t\right)&=ru_{i-3}\left(t\right)+\left(1-2r\right)u_{i-2}\left(t\right)+ru_{i-1}\left(t\right)\\
u_{i-1}\left(t+\Delta t\right)&=ru_{i-2}\left(t\right)+\left(1-2r\right)u_{i-1}\left(t\right)+ru_{i}\left(t\right)\\
u_{i}\left(t+\Delta t\right)&=ru_{i-1}\left(t\right)+\left(1-2r\right)u_{i}\left(t\right)+ru_{i+1}\left(t\right)\\
u_{i+1}\left(t+\Delta t\right)&=ru_{i}\left(t\right)+\left(1-2r\right)u_{i+1}\left(t\right)+ru_{i+2}\left(t\right)\\
u_{i+2}\left(t+\Delta t\right)&=ru_{i+1}\left(t\right)+\left(1-2r\right)u_{i+2}\left(t\right)+ru_{i+3}\left(t\right)
\end{aligned}
\tag{4.34}
$$

where

$$
r=\frac{\alpha\Delta t}{L^2}
\tag{4.35}
$$

Now suppose that the boundary conditions are that the nodal values are all zero. Suppose also that the algorithm begins with all nodes having a value of zero. In that case, one would expect the nodal values to remain at zero for all time. However, let us suppose that we perturb one of the nodes, node *i* say, so that it has a small non-zero value ε. From the nature of the problem, we would expect this nodal value to decay back towards the steady-state solution of zero, and this is what a stable solution will do. From Eqns. 4.33,

$$
\begin{aligned}
u_{i-2}\left(\Delta t\right)&=0 & u_{i-2}\left(2\Delta t\right)&=r^2\varepsilon\\
u_{i-1}\left(\Delta t\right)&=r\varepsilon & u_{i-1}\left(2\Delta t\right)&=2r\left(1-2r\right)\varepsilon\\
u_{i}\left(\Delta t\right)&=\left(1-2r\right)\varepsilon, & u_{i}\left(2\Delta t\right)&=\left[2r^2+\left(1-2r\right)^2\right]\varepsilon,\quad\ldots\\
u_{i+1}\left(\Delta t\right)&=r\varepsilon & u_{i+1}\left(2\Delta t\right)&=2r\left(1-2r\right)\varepsilon\\
u_{i+2}\left(\Delta t\right)&=0 & u_{i+2}\left(2\Delta t\right)&=r^2\varepsilon
\end{aligned}
\tag{4.36}
$$

As can be seen, the error becomes of the order $r^n\varepsilon$ at the nth time step, $t = n\Delta t$. Thus if $r > 1$, the initial small error will magnify without bound as time proceeds. If, on the other hand, $r \le 1$, the initial error will not grow. If $r < 1$, the error will diminish as time proceeds. Even if $r < 1$, there is still the possibility that the solution will oscillate in sign between negative and positive values, because of the $1 - 2r$ term; if $r < \frac{1}{2}$, the error will decrease without oscillation.

One says that the explicit Euler scheme with linear elements is unstable if $\Delta t > L^2 / \alpha$, and stable if

$$\Delta t < \frac{L^2}{\alpha} \tag{4.37}$$

The explicit scheme is **conditionally stable**, since it is only stable provided the time step is less some **critical time step** (or **stability limit**).

The above analysis was done for linear elements with a lumped mass matrix. A similar analysis can be carried our for any type of element or system. It is easier in the general case to examine the stability in terms of the eigenvalues of the system. It will be shown in §4.1.5 below that the Euler-Explicit scheme is more generally stable provided

Stability Requirement for Explicit-Euler:

$$\Delta t < \frac{2}{\lambda_{\max}} \tag{4.38}$$

where $\lambda_{\max}$ is the largest eigenvalue of the system. Further, the solution is non-oscillatory provided $\Delta t \le 1/\lambda_{\max}$. It seems that one has to evaluate the largest eigenvalue of the complete system to determine the critical time-step, but there is a powerful theorem of Linear Algebra which states that *the largest eigenvalue of an assembled system is less than the largest eigenvalue of any of the individual elements* in the model. Thus one need only determine the eigenvalues of the individual elements and use the maximum of these in the stability criterion.

It was mentioned above that as the mesh gets finer, $\lambda_{\max} = \mathrm{O}\left(1/h^2\right)$, where h is a mesh/element length parameter. This puts severe restrictions on the allowable time-step for very fine meshes.

Note the following:

- If one element of **K** is too large or one element of **C** is very small, then the maximum eigenvalue of the system will be increased and hence the critical time-step will be reduced. For this reason it is usual to *keep the FE mesh as uniform as possible*.
- If one uses higher order elements, the entries of **K** and **C** are more varied. It is usual to avoid this variation for the reason stated above, and hence it is typical to *use many lower-order elements* in an FE explicit analysis, rather than fewer higher-order elements.
- For the linear element, Eqn (4.23), $\lambda_{\max} = 12/L^2$. For the lumped **C** matrix one finds that $\lambda_{\max} = 4/L^2$, which allows for a larger time step.

2. Implicit Euler's method

In the implicit Euler scheme, approximate the derivative at time $t + \Delta t$ by the backward difference approximation

$$\dot{u}_{t+\Delta t} = \frac{u_{t+\Delta t} - u_t}{\Delta t} \tag{4.39}$$

In the implicit schemes, the FE equations are written at time $t + \Delta t$,

$$\mathbf{C}\dot{\mathbf{u}}(t+\Delta t) + \mathbf{K}\mathbf{u}(t+\Delta t) = \mathbf{F}(t+\Delta t) \tag{4.40}$$

and then rewritten as

Implicit Euler Algorithm:

$$\begin{aligned} &\overline{\mathbf{K}}\Delta\mathbf{u} = \mathbf{R} \\ &\quad \text{where } \overline{\mathbf{K}} = \mathbf{C} + \Delta t\mathbf{K} \\ &\qquad\quad \mathbf{R} = \Delta t\left[\mathbf{F}(t+\Delta t) - \mathbf{K}\mathbf{u}(t)\right] \\ &\mathbf{u}(t+\Delta t) = \mathbf{u}(t) + \Delta\mathbf{u} \end{aligned} \tag{4.41}$$

It can be shown that this scheme is stable provided $\Delta t\lambda_i \geq 0$. Thus the scheme is **unconditionally stable** provided the eigenvalues are all positive.

3. Semi-Implicit Euler's method

In the semi-implicit method, let

$$\dot{u}_{t+\alpha\Delta t} = \frac{u_{t+\Delta t} - u_t}{\Delta t} \tag{4.42}$$

with $0 \leq \alpha \leq 1$. This is equivalent to taking a linear variation of u between t and $t + \Delta t$, as illustrated below. The FE equations are now written at time $t + \alpha\Delta t$:

$$\mathbf{C}\dot{\mathbf{u}}(t + \alpha\Delta t) + \mathbf{K}\mathbf{u}(t + \alpha\Delta t) = \mathbf{F}(t + \alpha\Delta t) \tag{4.43}$$

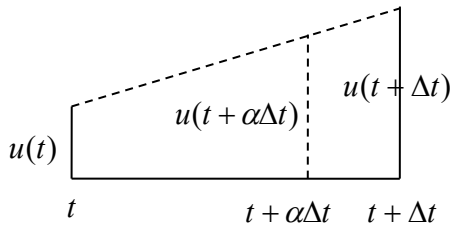

Figure 4.3: semi-implicit definition

Also,

$$\mathbf{u}(t + \alpha\Delta t) = \alpha\mathbf{u}(t + \Delta t) + (1 - \alpha)\mathbf{u}(t) \tag{4.44}$$

and the term $\mathbf{F}(t + \alpha\Delta t)$ is dealt with in a similar manner. This results in the scheme {▲Problem 3}

Semi-Implicit Euler Algorithm:

$$\overline{\mathbf{K}}\Delta\mathbf{u} = \mathbf{R}$$
$$\text{where } \overline{\mathbf{K}} = \mathbf{C} + \alpha\Delta t\mathbf{K}$$
$$\mathbf{R} = \Delta t\{\alpha\mathbf{F}(t+\Delta t) + (1-\alpha)\mathbf{F}(t) - \mathbf{K}\mathbf{u}(t)\} \tag{4.45}$$
$$\mathbf{u}(t+\Delta t) = \mathbf{u}(t) + \Delta\mathbf{u}$$

Note that for {▲Problem 4}

$\alpha = 0$	…	Explicit Euler	truncation error $0(\Delta t)$
$\alpha = \frac{1}{2}$	…	Crank-Nicholson scheme	truncation error $0(\Delta t^2)$
$\alpha = 1$	…	Implicit Euler	truncation error $0(\Delta t)$

For positive definite **C** and **K**, the stability criterion is $\Delta t \le 2/[(1-2\alpha)l_{\max}]$ for $0 \le \alpha < 0.5$; the scheme is unconditionally stable[4] for $\alpha \ge 0.5$. For a stable solution without numerical oscillation, the critical time step is half this value.

4. Predictor-Corrector method

In the predictor-corrector methods, one does the following:
(a) use an explicit formula to predict the first value of $\mathbf{u}(t+\Delta t)$
(b) use an implicit formula to improve that value by an iteration in place

4.1.4 Mode Superposition

A number of direct methods for the integration of the first order system (4.10) have been described above. An alternative solution procedure is the **mode superposition** method. The choice between these two methods is merely one of numerical effectiveness; the solutions obtained using either scheme are identical (if the same integration procedure is used in both). The mode superposition method has the advantage of providing information about the stability of the system (see later).

[4] by which is meant the scheme is stable for any time step. This does not mean that the scheme is accurate for large time-steps, merely that the solution will not diverge dramatically

The basic idea behind mode superposition is this: the FE equations $\mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$ are coupled equations, and to obtain a solution, all *n* equations need to be solved simultaneously. It is possible, however, to rewrite these equations in the form

$$\begin{aligned}
&\dot{z}^{(1)} + \lambda^{(1)} z^{(1)} = f^{(1)}, \qquad && f^{(1)} = \overline{u}_1^{(1)} F_1 + \overline{u}_2^{(1)} F_2 + \cdots \\
&\dot{z}^{(2)} + \lambda^{(2)} z^{(2)} = f^{(2)}, \qquad && f^{(2)} = \overline{u}_1^{(2)} F_1 + \overline{u}_2^{(2)} F_2 + \cdots \\
&\dot{z}^{(3)} + \lambda^{(3)} z^{(3)} = f^{(3)}, \qquad && f^{(3)} = \overline{u}_1^{(3)} F_1 + \overline{u}_2^{(3)} F_2 + \cdots \\
&\cdots
\end{aligned} \tag{4.46}$$

which are *n uncoupled* equations involving the *n* eigenvalues $\lambda^{(j)}$ and eigenvectors $\overline{u}^{(j)}$; each of these equations can be solved *independently* of the others. Once the equations have been solved for the so-called **generalised coordinates** $z^{(j)}$, u can be evaluated through (see below)

$$u_i = \sum_{j=1}^{n} \overline{u}_i^{(j)} z^{(j)} \tag{4.47}$$

that is, by summing up the contributions from all *n* eigenvectors/modes for that node.

The great advantage of the modal superposition method is that not all the equations need to be solved in order to obtain a solution. For example, one might solve the first three equations to obtain $z^{(1)}, z^{(2)}, z^{(3)}$ in which case

$$u_i \approx \overline{u}_i^{(1)} z^{(1)} + \overline{u}_i^{(2)} z^{(2)} + \overline{u}_i^{(3)} z^{(3)} \tag{4.48}$$

In other words, an approximate solution is found which only accounts for a limited number of modes, and it usually the first, limited, number of modes which dominate a solution.

The uncoupled differential equations (4.46) can be solved analytically when **F** is simple, for example when it is a constant or harmonic. For more complicated **F** the equations must be integrated using a numerical procedure, for example one of the direct numerical integration methods discussed earlier.

The modal equations in terms of the generalised coordinates are derived next. This is followed by a detailed example of the mode superposition method.

Derivation of the Modal Equations

Normalise the eigenvectors according to:

$$\overline{\mathbf{u}}^{(i)\mathrm{T}}\mathbf{C}\overline{\mathbf{u}}^{(j)} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \tag{4.49}$$

for $i, j = 1 \cdots n$, the number of degrees of freedom. Define the matrix $\mathbf{\Phi}$ whose columns are the eigenvectors $\overline{\mathbf{u}}^{(i)}$ and the diagonal matrix $\mathbf{\Omega}$ whose elements are the *n* eigenvalues:

$$\mathbf{\Phi} = \begin{bmatrix} \overline{u}_1^{(1)} & \overline{u}_1^{(2)} & \cdots & \overline{u}_1^{(n)} \\ \overline{u}_2^{(1)} & \overline{u}_2^{(2)} & \cdots & \overline{u}_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{u}_n^{(1)} & \overline{u}_n^{(2)} & \cdots & \overline{u}_n^{(n)} \end{bmatrix}, \qquad \mathbf{\Omega} = \begin{bmatrix} \lambda^{(1)} & 0 & \cdots & 0 \\ 0 & \lambda^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{(n)} \end{bmatrix} \tag{4.50}$$

To be clear, the subscripts here refer to the nodal locations, the superscripts refer to a particular mode. The *u* at any node *i* is a linear combination of the individual modal values for that node (*c.f.* Eqn. 4.27)

$$u_i(t) = \beta_1 \overline{u}_i^{(1)} \exp(-\lambda^{(1)} t) + \cdots + \beta_n \overline{u}_i^{(n)} \exp(-\lambda^{(n)} t), \qquad i = 1, 2, \cdots, n \tag{4.51}$$

The *n* solutions to the eigenvalue problem $\left[\mathbf{K} - \lambda^{(j)}\mathbf{C}\right]\overline{\mathbf{u}}^{(j)} = 0$ can be rewritten in the form

$$\mathbf{K\Phi} = \mathbf{C\Phi\Omega} \tag{4.52}$$

With the eigenvectors **C**-orthonormalised as in (4.49), one has $\mathbf{\Phi}^{\mathrm{T}}\mathbf{C\Phi} = \mathbf{I}$ and so, pre-multiplying the above equation by $\mathbf{\Phi}^{\mathrm{T}}$,

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{K\Phi} = \mathbf{\Omega} \tag{4.53}$$

Introduce now new generalised coordinates **z** such that

$$\mathbf{u}(t) = \mathbf{\Phi}\mathbf{z}(t), \qquad \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \overline{u}_1^{(1)} & \overline{u}_1^{(2)} & \cdots & \overline{u}_1^{(n)} \\ \overline{u}_2^{(1)} & \overline{u}_2^{(2)} & \cdots & \overline{u}_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{u}_n^{(1)} & \overline{u}_n^{(2)} & \cdots & \overline{u}_n^{(n)} \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \\ \vdots \\ z^{(n)} \end{bmatrix} \tag{4.54}$$

Then, pre-multiplying the equations (4.10) by $\mathbf{\Phi}^{\mathrm{T}}$ and using (4.54) gives

$$\begin{aligned} &\mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi}\dot{\mathbf{z}} + \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}\mathbf{z} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{F} \\ &\rightarrow \\ &\dot{\mathbf{z}} + \mathbf{\Omega}\mathbf{z} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{F} \end{aligned} \tag{4.55}$$

These equations are the uncoupled equations (4.46) given at the beginning of this subsection. Each equation can be integrated in turn to evaluate the coordinates $z^{(j)}$, whence the u_i can be evaluated through (4.54). For this purpose one needs the initial conditions on $\mathbf{z}(t)$. Since $\mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi} = \mathbf{I}$, then $\mathbf{u}(t) = \mathbf{\Phi}\,\mathbf{z}(t)$ becomes $\mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{u}(t) = \mathbf{z}(t)$ so that

$$\mathbf{z}(0) = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{u}(0) \tag{4.56}$$

Example

Consider the following problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad \text{B.C.} \quad \begin{matrix} u(0,t) = 0 \\ \partial u/\partial x(1,t) = 1 \end{matrix}, \quad \text{I.C.} \quad u(x,0) = \sin(-2.074x) \tag{4.57}$$

Using two linear elements, with $L = 1/2$, and applying the essential BC at $x = 0$, leads to the eigenvalues and eigenvectors:

$$\begin{aligned} &\mathbf{C} = \frac{1}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{K} = 2\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad \left|\mathbf{K} - \lambda\mathbf{C}\right| = 4 - \tfrac{5}{3}\lambda + \tfrac{7}{144}\lambda^2 = 0 \\ &\rightarrow \lambda^{(1)} = 2.597, \qquad \lambda^{(2)} = 31.689 \\ &\overline{\mathbf{u}}^{(1)} : \left[\mathbf{K} - 2.597\mathbf{C}\right]\overline{\mathbf{u}} = 0 \rightarrow \overline{\mathbf{u}}^{(1)} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} \\ &\overline{\mathbf{u}}^{(2)} : \left[\mathbf{K} - 31.689\mathbf{C}\right]\overline{\mathbf{u}} = 0 \rightarrow \overline{\mathbf{u}}^{(2)} = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \end{aligned} \tag{4.58}$$

Write the eigenvectors as

$$\overline{\mathbf{u}}^{(1)} = \eta^{(1)}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \qquad \overline{\mathbf{u}}^{(2)} = \eta^{(2)}\begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \tag{4.59}$$

with $\eta^{(1)}, \eta^{(2)}$ to be determined. Normalising according to (4.54) leads to four equations which can be used to obtain {▲Problem 5}

$$\eta^{(1)} = \sqrt{\frac{6}{4+\sqrt{2}}} \approx 1.053, \qquad \eta^{(2)} = \sqrt{\frac{6}{4-\sqrt{2}}} \approx 1.523 \qquad \text{and} \qquad \eta^{(1)}\eta^{(2)} = 0 \quad (4.60)$$

Form the matrices

$$\mathbf{\Phi} = \begin{bmatrix} 1.053 & 1.523 \\ 1.489 & -2.154 \end{bmatrix}, \quad \mathbf{\Omega} = \begin{bmatrix} 2.597 & 0 \\ 0 & 31.689 \end{bmatrix} \tag{4.61}$$

With the natural BC at $x = 1$, constant over time, $\partial u / \partial x(1,t) = 1$, the **F** vector is

$$\mathbf{F} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.62}$$

The modal equations are

$$\begin{aligned} &\dot{\mathbf{z}} + \mathbf{\Omega z} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{F} \\ &\rightarrow \begin{bmatrix} \dot{z}^{(1)} \\ \dot{z}^{(2)} \end{bmatrix} + \begin{bmatrix} 2.597 & 0 \\ 0 & 31.689 \end{bmatrix} \begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix} = \begin{bmatrix} 1.053 & 1.523 \\ 1.489 & -2.154 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{4.63}$$

or

$$\begin{aligned} \dot{z}^{(1)} + 2.597 z^{(1)} &= 1.523 \\ \dot{z}^{(2)} + 31.689 z^{(1)} &= -2.154 \end{aligned} \tag{4.64}$$

These are first order ODEs which can be solved for $z^{(i)}$:

$$\begin{aligned} z^{(1)} &= +0.586 + Ae^{-2.597t} \\ z^{(2)} &= -0.068 + Be^{-31.689t} \end{aligned} \tag{4.65}$$

From the initial condition $u(x,0) = \sin(-2.074x)$:

$$\mathbf{z}(0)=\begin{bmatrix} z^{(1)}(0) \\ z^{(2)}(0) \end{bmatrix}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{u}(0)=\begin{bmatrix} 1.053 & 1.489 \\ 1.523 & -2.154 \end{bmatrix}\frac{1}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u_1(\frac{1}{2},0) \\ u_2(1,0) \end{bmatrix}$$
$$=\begin{bmatrix} 0.475 & 0.336 \\ 0.328 & -0.232 \end{bmatrix}\begin{bmatrix} -0.861 \\ -0.876 \end{bmatrix} \tag{4.66}$$
$$=\begin{bmatrix} -0.703 \\ -0.079 \end{bmatrix}$$

Using these initial conditions leads to evaluation of the constants A and B:

$$\begin{aligned} z^{(1)} &= +0.586-1.289e^{-2.597t} \\ z^{(2)} &= -0.068-0.011e^{-31.689t} \end{aligned} \tag{4.67}$$

Finally, the values of u_i are obtained through $\mathbf{u}(t)=\mathbf{\Phi}\,\mathbf{z}(t)$:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=\begin{bmatrix} 1.053 & 1.523 \\ 1.489 & -2.154 \end{bmatrix}\begin{bmatrix} z^{(1)} \\ z^{(2)} \end{bmatrix}=\begin{bmatrix} 0.513-1.357e^{-2.597t}-0.017e^{-31.689t} \\ 1.019-1.919e^{-2.597t}+0.024e^{-31.689t} \end{bmatrix} \tag{4.68}$$

4.1.5 Stability

In the following, the stability criterion Eqn. 4.38 for the Explicit Euler scheme, Eqn. 4.27, is derived (but the same methods may be used to analyse any numerical integration scheme).

The mode superposition and direct integration methods both involve the integration of differential equations. They are two slightly different ways of solving the same problem; if one supposes that an FE problem is solved using both methods, and (4.10, 4.46) are both solved using the same numerical scheme, each with the same time step Δt, *both methods are completely equivalent*. Therefore, to study the accuracy of direct integration, one may focus on and estimate the accuracy and stability of integration of the modal equations, $\dot{\mathbf{z}}+\mathbf{\Omega}\mathbf{z}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{F}$, which is an easier task. Furthermore, since all the modal equations are similar one need only examine one typical equation, which may be written as

$$\dot{z}+\lambda z=f \tag{4.69}$$

In fact, this is just the one-dimensional equation considered earlier, Eqn. 4.28, and the explicit Euler scheme was examined in relation to this equation in Eqns. 4.29-4.30. Nevertheless, although the following is repetition to a large extent, we will examine it again anew in the current context.

Stability is determined by examining the numerical solution for arbitrary initial conditions. One may consider the case of $f = 0$, and one sees that the stability and accuracy depends on the eigenvalue λ and whatever time-step is used. Thus, considering the homogeneous modal equation

$$\dot{z} + \lambda z = 0 \tag{4.70}$$

Separating variables and solving gives the general solution

$$z = Ae^{-\lambda t} \tag{4.71}$$

Starting with initial condition $z(t)$ at time t, one has $z(t+\Delta t) = z(t)\exp\{-\lambda \Delta t\}$. Regardless of the time-stepping algorithm used, then, one requires for a stable solution:

$$\begin{aligned} |z(t+\Delta t)| &< |z(t)| \qquad \lambda > 0 \\ z(t+\Delta t) &= z(t) \qquad \lambda = 0 \end{aligned} \tag{4.72}$$

with instability for $\lambda < 0$.

Examining now the explicit Euler scheme, replace the $\dot{z}$ in Eqn. 4.70 with $\left[z(t+\Delta t) - z(t)\right]/\Delta t$, leading to

$$z(t+\Delta t) = A\, z(t) \tag{4.73}$$

where A is the **amplification factor** (so called, since any errors at one time step will be magnified by this amount into the next time step)

$$A = 1 - \lambda \Delta t \tag{4.74}$$

When $\lambda = 0$, $z(t+\Delta t) = z(t)$ as before and, when $\lambda > 0$, in order that $|z(t+\Delta t)| < |z(t)|$, it is required that $|A| < 1$, or $-1 < 1 - \lambda \Delta t < 1$. The inequality on the right is always satisfied; the left-hand inequality leads to the condition

$$\Delta t < \frac{2}{\lambda} \tag{4.75}$$

This stability condition must hold for all modes in the system. The largest eigenvalue $\lambda_{\max}$ imposes the greatest restriction, leading to the criterion (4.38).

4.2 Second-Order Systems

Here, the hyperbolic second order system (4.4) is examined. In particular, consider the following problem:

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad \text{B.C.} \quad \begin{matrix} u(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(l,t) = 0 \end{matrix}, \qquad \text{I.C.} \quad \begin{matrix} u(x,0) = 0 \\ \dfrac{\partial u}{\partial t}(x,0) = \dfrac{2x}{l} \end{matrix} \tag{4.76}$$

4.2.1 FE equations for Second Order Systems

Using the Galerkin procedure to discretise the space variable, one has for a linear element,

$$\frac{\partial^2 u_i}{\partial t^2} \int_{x_i}^{x_{i+1}} N_1 N_j dx + \frac{\partial^2 u_{i+1}}{\partial t^2} \int_{x_i}^{x_{i+1}} N_2 N_j dx + u_i c^2 \int_{x_i}^{x_{i+1}} \frac{\partial N_1}{\partial x}\frac{\partial N_j}{\partial x} dx + u_{i+1} c^2 \int_{x_i}^{x_{i+1}} \frac{\partial N_2}{\partial x}\frac{\partial N_j}{\partial x} dx = c^2 \left[\frac{\partial u}{\partial x} N_j \right]_{x_i}^{x_{i+1}}$$

$$j = 1,2 \tag{4.77}$$

Evaluating the integrals leads to the element equations[5]

$$\frac{L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_i \\ \ddot{u}_{i+1} \end{bmatrix} + c^2 \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_i \\ u_{i+1} \end{bmatrix} = c^2 \begin{bmatrix} -u'(x_i) \\ +u'(x_{i+1}) \end{bmatrix} \tag{4.78}$$

element mass matrix **M_e**

element stiffness matrix **K_e**

After assembly, one has the system of second order ODEs of the form

[5] the first matrix here is called the *mass* matrix, so-called because of its physical relevance in elastodynamic problems (see later)

$$\frac{L}{6}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \vdots \\ \ddot{u}_{n+1} \end{bmatrix} + \frac{c^2}{L}\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n+1} \end{bmatrix} = c^2\begin{bmatrix} -u'(0) \\ 0 \\ 0 \\ \vdots \\ +u'(l) \end{bmatrix} \tag{4.79}$$

or, in short,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F} \tag{4.80}$$

which can be solved in a number of different ways (see below).

4.2.2 Eigenvalues and Eigenvectors

As with the first order system, an eignenvalue analysis can be carried out for the system $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{F}$, and will tell us much useful information. First, consider the single degree of freedom model, $m\ddot{u} + ku = f$, with f constant (see the Appendix to his Chapter), which has the solution

$$u(t) = A\sin(\omega t + \phi) + \frac{f}{k} \tag{4.81}$$

This is an oscillation at **natural frequency** ω about the mean position f/k (which is the solution to the time independent "static" equation $ku = f$). A solution can be obtained for the complete system by assuming that it also oscillates about some mean configuration $\mathbf{u}_m = \mathbf{K}^{-1}\mathbf{F}$,

$$\mathbf{u}(t) = \mathbf{u}_m + \bar{\mathbf{u}}\sin(\omega t + \phi) \tag{4.82}$$

Substitution into the FE equations (4.76) gives

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\bar{\mathbf{u}} = 0 \tag{4.83}$$

and this system of $n \times n$ equations in the n entries of $\bar{\mathbf{u}} = \begin{bmatrix}\bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n\end{bmatrix}^{\mathrm{T}}$ has a solution only if the determinant of the coefficient matrix is zero:

$$\left|\mathbf{K}-\omega^2\mathbf{M}\right|=0 \tag{4.84}$$

This equation can be solved for the *n* eigenvalues ω^2.

Using two linear elements for the example problem (4.76), applying the boundary condition $u_1=u(0,t)=0$ ($\ddot{u}_1=\ddot{u}(0,t)=0$), and eliminating the first row and column, leads to the equations

$$\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=\mathbf{F},\qquad \mathbf{M}=\frac{l}{12}\begin{bmatrix}4 & 1\\ 1 & 2\end{bmatrix},\quad \mathbf{u}=\begin{bmatrix}u_2\\ u_3\end{bmatrix},\quad \mathbf{K}=\frac{2c^2}{l}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{4.85}$$

where $l=2L$ and so

$$\left|\mathbf{K}-\omega^2\mathbf{M}\right|=\mathbf{0}\quad\rightarrow\quad\frac{2c^2}{l}\begin{vmatrix}2-4\alpha & -1-\alpha\\ -1-\alpha & 1-2\alpha\end{vmatrix}=0,\quad \alpha=\frac{\omega^2l^2}{24c^2} \tag{4.86}$$

Thus $7\alpha^2-10\alpha+1=0$ which yields $\alpha=\left(5\pm\sqrt{18}\right)/7$ and the square-roots of the eigenvalues, the natural frequencies, are then

$$\begin{aligned}\omega^{(1)}&=1.61142\frac{c}{l},\quad \omega^{(2)}=5.62930\frac{c}{l}\\ &=0.8057\frac{c}{L},\qquad\quad =2.8147\frac{c}{L}\end{aligned} \tag{4.87}$$

Note that the same eigenvalues would be obtained from the 3×3 global system (after applying the essential BC but *not* eliminating a row and column); the third eigenvalue would be $\omega=1$.

This eigenvalue analysis will be continued further below in section 4.2.5, in the context of the elastodynamic problem, where the eigenvalues and eigenvectors have a specific physical meaning.

4.2.3 Direct Integration for Second Order Systems

As with first order systems, one can solve (4.80) using either one of many direct integration methods or through mode superposition. The direct integration methods are discussed here.

As with first order systems, a number of different direct integration methods are available for the integration of the second order system (4.79), for example,

1. Explicit Central Difference Scheme
2. Linear Acceleration Scheme (Implicit)
3. Wilson θ Scheme (Implicit)
4. Newmark Scheme (Implicit)
5. Trapezoidal Scheme (Implicit)

1. Explicit Central Difference Scheme

In the explicit central difference scheme, one expands the unknown nodal functions $u_i(t)$ in Taylor series:

$$\begin{aligned} u_i(t+\Delta t) &= u_i(t) + \Delta t\dot{u}_i(t) + \frac{1}{2}(\Delta t)^2\ddot{u}_i(t) + \cdots \\ u_i(t-\Delta t) &= u_i(t) - \Delta t\dot{u}_i(t) + \frac{1}{2}(\Delta t)^2\ddot{u}_i(t) + \cdots \end{aligned} \tag{4.88}$$

Adding and subtracting these expressions then lead to the following approximations for the derivatives:

$$\begin{aligned} \ddot{u}_i(t) &= \frac{u_i(t+\Delta t) - 2u_i(t) + u_i(t-\Delta t)}{(\Delta t)^2} \\ \dot{u}_i(t) &= \frac{u_i(t+\Delta t) - u_i(t-\Delta t)}{2\Delta t} \end{aligned} \tag{4.89}$$

Being an explicit scheme, the FE equations are considered at time *t*,

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \tag{4.90}$$

Substituting in the approximate expressions for the derivatives leads to

Explicit Central Difference Scheme:

$$\left(\frac{1}{(\Delta t)^2}\mathbf{M}\right)\mathbf{u}(t+\Delta t) = \hat{\mathbf{F}}(t)$$
$$\hat{\mathbf{F}}(t) = \mathbf{F}(t) - \left(\mathbf{K} - \frac{2}{(\Delta t)^2}\mathbf{M}\right)\mathbf{u}(t) - \left(\frac{1}{(\Delta t)^2}\mathbf{M}\right)\mathbf{u}(t-\Delta t) \tag{4.91}$$

To start the scheme, one needs the value of $\mathbf{u}(-\Delta t)$. To obtain this value, note that $\mathbf{u}(0), \dot{\mathbf{u}}(0)$ are known from the initial conditions, and one can hence obtain $\ddot{\mathbf{u}}(0)$ from the equations $\mathbf{M}\ddot{\mathbf{u}}(0) + \mathbf{K}\mathbf{u}(0) = \mathbf{F}(0)$. One can then re-arrange the approximate expressions for $\dot{\mathbf{u}}(t), \ddot{\mathbf{u}}(t)$ above to obtain

$$\begin{aligned}\mathbf{u}(-\Delta t) &= \mathbf{u}(0) - \Delta t\dot{\mathbf{u}}(0) + \frac{1}{2}(\Delta t)^2\ddot{\mathbf{u}}(0) \\ &= \mathbf{u}(0) - \Delta t\dot{\mathbf{u}}(0) + \frac{1}{2}(\Delta t)^2\mathbf{M}^{-1}\left[\mathbf{F}(0) - \mathbf{K}\mathbf{u}(0)\right]\end{aligned} \tag{4.92}$$

Considering a one-dimensional case for illustrative purposes, consider the ODE

$$\frac{d^2u}{dt^2} + \omega^2 u = f, \quad \dot{u}(0) = \overline{v}_0, \quad u(0) = \overline{u}_0 \tag{4.93}$$

Substituting $\ddot{u}_t = \left[u_{t+\Delta t} - 2u_t + u_{t-\Delta t}\right]/(\Delta t)^2$ into Eqn. 4.93 gives

$$u(t+\Delta t) = \left[2 - \omega^2(\Delta t)^2\right]u(t) - u(t-\Delta t) + f(\Delta t)^2 \tag{4.94}$$

It is convenient to express the relationship between the values at the different time-steps in the form of the matrix recursive algorithm:

$$\begin{bmatrix} u(t+\Delta t) \\ u(t) \end{bmatrix} = \begin{bmatrix} 2-\omega^2\Delta t^2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u(t) \\ u(t-\Delta t) \end{bmatrix} + \begin{bmatrix} \Delta t^2 \\ 0 \end{bmatrix} f(t) \tag{4.95}$$

To keep things simple, let $f = 0$, so that at any time $t = n\Delta t$, the solution is given by

$$\begin{bmatrix} u(n\Delta t) \\ u((n-1)\Delta t) \end{bmatrix} = \mathbf{A}\begin{bmatrix} u((n-1)\Delta t) \\ u((n-2)\Delta t) \end{bmatrix} = \mathbf{A}^n \begin{bmatrix} u(0) \\ u(-\Delta t) \end{bmatrix} \tag{4.96}$$

where

$$\mathbf{A} = \begin{bmatrix} 2-\omega^2\Delta t^2 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.97}$$

The value of $u(-\Delta t)$ can be obtained in the same way as was Eqn. 4.92:

$$u(-\Delta t) = \left[1-\tfrac{1}{2}\omega^2(\Delta t)^2\right]u(0) - \Delta t\dot{u}(0) \tag{4.98}$$

This solution is plotted in Fig. 4.4 for $\omega = 2$, $\dot{u}(0) = 3/2$; for $\Delta t = 0.5$ and $\Delta t = 1.05$, together with the exact solution $u(t) = \frac{3}{4}\sin(2t)$. The solution is quite accurate for $\Delta t = 0.5$. On the other hand, when the time step is as large as $\Delta t = 1$, the solution is highly inaccurate; this issue is discussed further below.

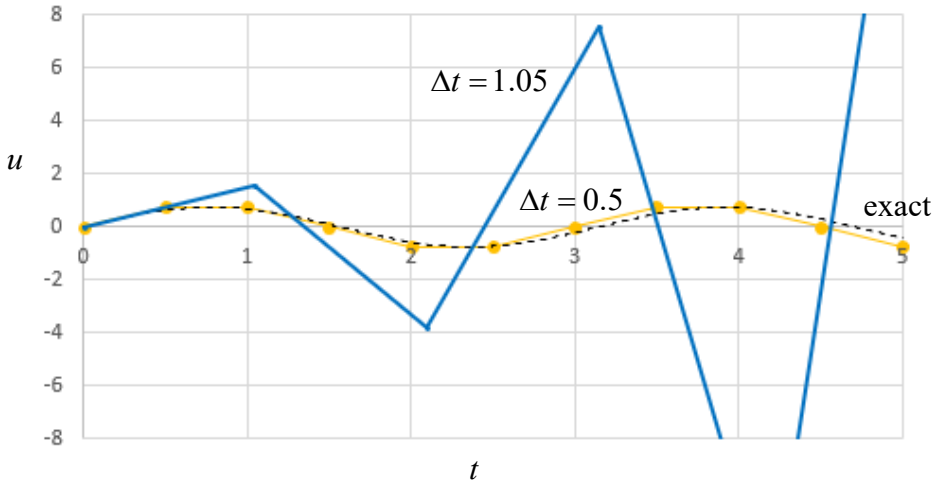

Figure 4.4: Explicit Central Difference scheme for the solution of an ODE

Matrix Lumping

Analogous to the first order system, the inversion of the explicit central difference equations can be greatly speeded up by having **M** diagonal, that is by lumping the **M** matrix.

Stability

Examining the one-dimensional case, Eqn. 4.93, $\ddot{u} + \omega^2 u = f$, the general solution is periodic in form. In the case of constant *f*, the solution is

$$u(t) = \left[u(0) - f/\omega^2\right]\cos(\omega t) + \left[\dot{u}(0)/\omega\right]\sin(\omega t) + f/\omega^2 \tag{4.99}$$

The solution is seen to oscillate about $u = f/\omega^2$. The explicit Central Difference numerical approximation of this exact solution is, on the other hand, given by Eqn. 4.95. In the case of $f = 0$, it is given by Eqn. 4.96. The matrix **A**, Eqn. 4.97, is clearly critical to the stability of the numerical scheme. It is helpful now to decompose the matrix **A** into its **eigendecomposition** (**spectral decomposition**): $\mathbf{A} = \mathbf{PJP}^{-1}$. Here, **J** is the diagonal matrix of eigenvalues and **P** is the matrix of eigenvectors (columns of **P** are the eigenvectors). This decomposition has the special property that $\mathbf{A}^n = \mathbf{PJ}^n\mathbf{P}^{-1}$. Evaluating the eigenvalues and eigenvectors of **A**, one has

$$\mathbf{A} = \begin{bmatrix} 2-\alpha & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2-\alpha}{2} + i\Delta & \frac{2-\alpha}{2} - i\Delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2-\alpha}{2} + i\Delta & 0 \\ 0 & \frac{2-\alpha}{2} - i\Delta \end{bmatrix} \begin{bmatrix} -i\frac{1}{2\Delta} & \frac{1}{2} + i\frac{2-\alpha}{4\Delta} \\ +i\frac{1}{2\Delta} & \frac{1}{2} - i\frac{2-\alpha}{4\Delta} \end{bmatrix} \tag{4.100}$$

where

$$\Delta = \sqrt{1 - \frac{(2-\alpha)^2}{4}}, \qquad \alpha = \omega^2(\Delta t)^2 \tag{4.101}$$

Thus, from Eqn. 4.96,

$$\begin{bmatrix} u(n\Delta t) \\ u((n-1)\Delta t) \end{bmatrix} = \begin{bmatrix} \frac{2-\alpha}{2} + i\Delta & \frac{2-\alpha}{2} - i\Delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{2-\alpha}{2} + i\Delta & 0 \\ 0 & \frac{2-\alpha}{2} - i\Delta \end{bmatrix}^n \begin{bmatrix} -i\frac{1}{2\Delta} & \frac{1}{2} + i\frac{2-\alpha}{4\Delta} \\ +i\frac{1}{2\Delta} & \frac{1}{2} - i\frac{2-\alpha}{4\Delta} \end{bmatrix} \begin{bmatrix} u(0) \\ u(-\Delta t) \end{bmatrix} \tag{4.102}$$

The eigenvalues are complex for $\alpha < 4$. In fact, for $\alpha < 4$, the absolute value of the eigenvalues is always 1:

$$\left| \frac{2-\alpha}{2} \pm i\sqrt{1 - \frac{(2-\alpha)^2}{4}} \right| = 1 \tag{4.103}$$

in which case $\mathbf{A}^n$ is bounded as n increases. In the case of $\alpha > 4$, the eigenvalues are real and the magnitude is always greater than 1, in which case $\mathbf{A}^n$ becomes unbounded. The criterion for stability is therefore that $\alpha < 4$, or

$$\Delta t < \frac{2}{\omega} \tag{4.104}$$

This criterion is seen to be satisfied in Fig. 4.4, for which $\omega = 2$, so that the stability criterion is $\Delta t < 1$.

More generally, as with the explicit Euler scheme for the first order system discussed in section 4.1.3, the stability of the explicit Central Difference scheme can be examined by considering the system of equations for a general differential equation of the form 4.76. The FE equations for various internal nodes in the mesh resulting from the use of linear elements and a lumped mass matrix are:

$$\begin{aligned}
\ddot{u}_{i-2} + \frac{c^2}{L^2}\{-u_{i-3} + 2u_{i-2} - u_{i-1}\} &= 0 \\
\ddot{u}_{i-1} + \frac{c^2}{L^2}\{-u_{i-2} + 2u_{i-1} - u_i\} &= 0 \\
\ddot{u}_i + \frac{c^2}{L^2}\{-u_{i-1} + 2u_i - u_{i+1}\} &= 0 \\
\ddot{u}_{i+1} + \frac{c^2}{L^2}\{-u_i + 2u_{i+1} - u_{i+2}\} &= 0 \\
\ddot{u}_{i+2} + \frac{c^2}{L^2}\{-u_{i+1} + 2u_{i+2} - u_{i+3}\} &= 0
\end{aligned} \tag{4.105}$$

Using the Central Difference approximation, Eqn. 4.89,

$$
\begin{aligned}
u_{i-2}(t+\Delta t) &= -u_{i-2}(t-\Delta t) + ru_{i-3}(t) + 2(1-r)u_{i-2}(t) + ru_{i-1}(t) \\
u_{i-1}(t+\Delta t) &= -u_{i-1}(t-\Delta t) + ru_{i-2}(t) + 2(1-r)u_{i-1}(t) + ru_{i}(t) \\
u_{i}(t+\Delta t) &= -u_{i}(t-\Delta t) + ru_{i-1}(t) + 2(1-r)u_{i}(t) + ru_{i+1}(t) \\
u_{i+1}(t+\Delta t) &= -u_{i+1}(t-\Delta t) + ru_{i}(t) + 2(1-r)u_{i+1}(t) + ru_{i+2}(t) \\
u_{i+2}(t+\Delta t) &= -u_{i+2}(t-\Delta t) + ru_{i+1}(t) + 2(1-r)u_{i+2}(t) + ru_{i+3}(t)
\end{aligned}
\tag{4.106}
$$

where

$$
r = \frac{c^2(\Delta t)^2}{L^2} \tag{4.107}
$$

Suppose now that we begin the algorithm with all nodal values and initial conditions zero except for node *i*, which is given a small non-zero value ε. From Eqns. 4.106,

$$
\begin{array}{lll}
u_{i-2}(\Delta t) = 0 & u_{i-2}(2\Delta t) = r^2\varepsilon & u_{i-2}(3\Delta t) = \left(-6r^3 + 6r^2\right)\varepsilon \\
u_{i-1}(\Delta t) = r\varepsilon & u_{i-1}(2\Delta t) = \left(-4r^2 + 4r\right)\varepsilon & u_{i-1}(3\Delta t) = \left(15r^3 - 24r^2 + 10r\right)\varepsilon \\
u_{i}(\Delta t) = 2(1-r)\varepsilon, & u_{i}(2\Delta t) = \left(6r^2 - 8r + 3\right)\varepsilon, & u_{i}(3\Delta t) = \left(-28r^3 + 44r^2 - 12r - 4\right)\varepsilon \\
u_{i+1}(\Delta t) = r\varepsilon & u_{i+1}(2\Delta t) = \left(-4r^2 + 4r\right)\varepsilon & u_{i+1}(3\Delta t) = \left(15r^3 - 24r^2 + 10r\right)\varepsilon \\
u_{i+2}(\Delta t) = 0 & u_{i+2}(2\Delta t) = r^2\varepsilon & u_{i+2}(3\Delta t) = \left(-6r^3 + 6r^2\right)\varepsilon
\end{array}
\tag{4.108}
$$

As can be seen, the error becomes of the order $r^n\varepsilon$ at the *n*th time step, $t = n\Delta t$. In the same way as with the explicit Euler scheme earlier, it can be seen that the explicit Central Difference scheme for linear elements is conditionally stable, with stability for

$$
\Delta t < \frac{L}{c} \tag{4.109}
$$

The above analysis was done for linear elements with a lumped mass matrix. More generally, as proved below, the Explicit Central Difference scheme is stable provided

Stability Requirement for Explicit Central Difference Scheme:

$$\Delta t < \frac{2}{\omega_{\max}} \tag{4.110}$$

where $\omega_{\max}$ is the largest natural frequency of the system. As for the explicit Euler scheme, one need only determine the natural frequency of the individual elements and use the maximum of these in the stability criterion.

For the same reasons given regarding the first order system, when using the Central Difference explicit scheme, the mesh should be kept as regular as possible and low-order (linear) elements should be used if possible.

2. Linear Acceleration Scheme (Implicit)

Here, suppose that the quantities $u, \dot{u}, \ddot{u}$ at time t are known. To find the values of these quantities a time Δt later, assume a linearly varying "acceleration"[6] $\ddot{u}$ in the time step. Let τ be the increase in time starting at time t. From Fig. 4.5,

$$\ddot{u}(t+\tau) = \ddot{u}(t) + \frac{\tau}{\Delta t}\left(\ddot{u}(t+\Delta t) - \ddot{u}(t)\right) \tag{4.11}$$

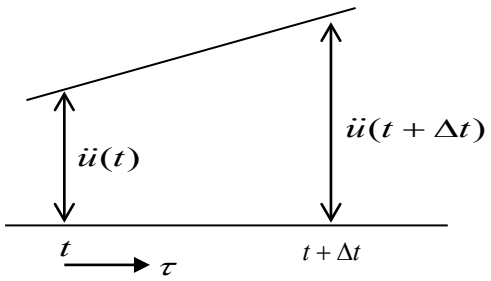

Figure 4.5: Linear Acceleration Scheme

Integration with respect to τ then gives (the $\dot{u}(t), u(t)$ terms are constants of integration)

[6] this terminology assumes that u represents a "displacement"

$$\begin{aligned}\dot{u}(t+\tau) &= \dot{u}(t) + \tau\ddot{u}(t) + \frac{\tau^2}{2\Delta t}\left(\ddot{u}(t+\Delta t) - \ddot{u}(t)\right) \\ u(t+\tau) &= u(t) + \tau\dot{u}(t) + \frac{\tau^2}{2}\ddot{u}(t) + \frac{\tau^3}{6\Delta t}\left(\ddot{u}(t+\Delta t) - \ddot{u}(t)\right)\end{aligned} \tag{4.112}$$

Substituting $\tau = \Delta t$ into these equations and rearranging gives

$$\begin{aligned}\dot{u}(t+\Delta t) &= \dot{u}(t) + \frac{\Delta t}{2}\left(\ddot{u}(t+\Delta t) + \ddot{u}(t)\right) \\ \ddot{u}(t+\Delta t) &= \frac{6}{(\Delta t)^2}\left(u(t+\Delta t) - u(t)\right) - \frac{6}{\Delta t}\dot{u}(t) - 2\ddot{u}(t)\end{aligned} \tag{4.113}$$

Substituting the latter equation into the former finally leads to the expressions

$$\begin{aligned}\dot{u}(t+\Delta t) &= \frac{3}{\Delta t}\left(u(t+\Delta t) - u(t)\right) - 2\dot{u}(t) - \frac{\Delta t}{2}\ddot{u}(t) \\ \ddot{u}(t+\Delta t) &= \frac{6}{(\Delta t)^2}\left(u(t+\Delta t) - u(t)\right)\frac{6}{\Delta t}\dot{u}(t) - 2\ddot{u}(t)\end{aligned} \tag{4.114}$$

The FE equations 4.80 are written at time $t + \Delta t$,

$$\mathbf{M}\ddot{\mathbf{u}}(t+\Delta t) + \mathbf{K}\mathbf{u}(t+\Delta t) = \mathbf{F}(t+\Delta t) \tag{4.115}$$

The use of Eqn. 4.114 then leads to

Linear Acceleration Scheme:

$$\left[\mathbf{M}\frac{6}{(\Delta t)^2} + \mathbf{K}\right]\mathbf{u}(t+\Delta t) = \mathbf{F}(t+\Delta t) + \mathbf{M}\left[\frac{6}{(\Delta t)^2}\mathbf{u}(t) + \frac{6}{\Delta t}\dot{\mathbf{u}}(t) + 2\ddot{\mathbf{u}}(t)\right] \tag{4.116}$$

Once $\mathbf{u}(t+\Delta t)$ is obtained, then $\dot{\mathbf{u}}(t+\Delta t)$ and $\ddot{\mathbf{u}}(t+\Delta t)$ can be obtained from (4.116). The scheme is unconditionally stable.

3. The Wilson θ Scheme (Implicit)

Here, assume a linearly varying $\ddot{u}$ in the time step, but now extrapolate the hypothetical solution out to time $t+\theta\Delta t$, where θ is some parameter ($\theta \geq 1$), as illustrated in Fig. 4.6. When $\theta = 1$ the method reduces to the linear acceleration scheme. Then

$$\ddot{u}(t+\tau) = \ddot{u}(t) + \frac{\tau}{\theta\Delta t}\left(\ddot{u}(t+\theta\Delta t) - \ddot{u}(t)\right) \tag{4.117}$$

Integration with respect to τ, the substitution $\tau = \theta\Delta t$, and some rearranging then leads to {▲Problem 9}

$$\begin{aligned}
\dot{u}(t+\theta\Delta t) &= \frac{3}{\theta\Delta t}\left(u(t+\theta\Delta t) - u(t)\right) - 2\dot{u}(t) - \frac{\theta\Delta t}{2}\ddot{u}(t) \\
\ddot{u}(t+\theta\Delta t) &= \frac{6}{(\theta\Delta t)^2}\left(u(t+\theta\Delta t) - u(t)\right) - \frac{6}{\theta\Delta t}\dot{u}(t) - 2\ddot{u}(t)
\end{aligned} \tag{4.118}$$

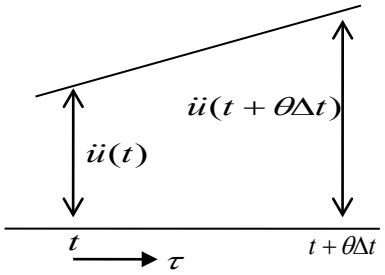

Figure 4.6: The Wilson θ Scheme

The FE equations are now written at time $t+\theta\Delta t$:

$$\mathbf{M}\ddot{\mathbf{u}}(t+\theta\Delta t) + \mathbf{K}\mathbf{u}(t+\theta\Delta t) = \mathbf{F}(t+\theta\Delta t) \tag{4.119}$$

As with the linearly varying $\ddot{u}$, the $\mathbf{F}$ vector is approximated by $\overline{\mathbf{F}}$, a linear extrapolation:

$$\overline{\mathbf{F}}(t+\theta\Delta t) = \mathbf{F}(t) + \theta\left(\mathbf{F}(t+\Delta t) - \mathbf{F}(t)\right) \tag{4.120}$$

These expressions lead to

Wilson θ Scheme:

$$\left[\mathbf{M}\frac{6}{(\theta\Delta t)^2}+\mathbf{K}\right]\mathbf{u}(t+\theta\Delta t)=\overline{\mathbf{F}}(t+\theta\Delta t)+\mathbf{M}\left[\frac{6}{(\theta\Delta t)^2}\mathbf{u}(t)+\frac{6}{\theta\Delta t}\dot{\mathbf{u}}(t)+2\ddot{\mathbf{u}}(t)\right] \quad (4.121)$$

To obtain the solution at time $t+\Delta t$, the solution for $\mathbf{u}(t+\theta\Delta t)$ is substituted into (4.116b). This is then used in (4.117) and its two integrated equations, and τ is set to Δt. This leads to

$$\begin{aligned}
\ddot{\mathbf{u}}(t+\Delta t)&=\frac{6}{\theta(\theta\Delta t)^2}\left(\mathbf{u}(t+\theta\Delta t)-\mathbf{u}(t)\right)-\frac{6}{\theta(\theta\Delta t)}\dot{\mathbf{u}}(t)+\left(1-\frac{3}{\theta}\right)\ddot{\mathbf{u}}(t)\\
\dot{\mathbf{u}}(t+\Delta t)&=\dot{\mathbf{u}}(t)+\frac{\Delta t}{2}\left(\ddot{\mathbf{u}}(t+\Delta t)+\ddot{\mathbf{u}}(t)\right)\\
\mathbf{u}(t+\Delta t)&=\mathbf{u}(t)+\Delta t\dot{\mathbf{u}}(t)+\frac{(\Delta t)^2}{6}\left(\ddot{\mathbf{u}}(t+\Delta t)+2\ddot{\mathbf{u}}(t)\right)
\end{aligned} \quad (4.122)$$

This scheme is unconditionally stable for $\theta\geq 1.37$.

4. Newmark Scheme (Implicit)

In the Newmark integration scheme, first expand as a Taylor series

$$u(t+\Delta t)=u(t)+\Delta t\dot{u}(t)+\frac{1}{2}(\Delta t)^2\ddot{u}(t)+\frac{1}{6}(\Delta t)^3\dddot{u}(\xi),\qquad t\leq\xi\leq t+\Delta t \quad (4.123)$$

Using a linear approximation for the $\ddot{u}$ term,

$$\begin{aligned}
\ddot{u}(t+\Delta t)&=\ddot{u}(t)+\Delta t\dddot{u}(t)\quad+O(\Delta t)^2\\
&=\ddot{u}(t)+\Delta t\dddot{u}(\xi)
\end{aligned} \quad (4.124)$$

Thus the error term in the Taylor series can be written in terms of some parameter α:

$$\begin{aligned}
u(t+\Delta t)&=u(t)+\Delta t\dot{u}(t)+\frac{1}{2}(\Delta t)^2\ddot{u}(t)+\alpha(\Delta t)^2\left(\ddot{u}(t+\Delta t)-\ddot{u}(t)\right)\\
&=u(t)+\Delta t\dot{u}(t)+(\Delta t)^2\left(\alpha\ddot{u}(t+\Delta t)+(\tfrac{1}{2}-\alpha)\ddot{u}(t)\right)
\end{aligned} \quad (4.125)$$

When $\alpha = 1/6$ the linear acceleration scheme expression (4.112b) is recovered. Similarly, the following assumption is made regarding the $\dot{u}$ term:

$$\dot{u}(t+\Delta t) = \dot{u}(t) + \Delta t\left(\delta\ddot{u}(t+\Delta t) + (1-\delta)\ddot{u}(t)\right) \tag{4.126}$$

When $\delta = 1/2$, the expression (4.114a) from the linear acceleration scheme is recovered.

Solving (4.125) for $\ddot{\mathbf{u}}(t+\Delta t)$ and substituting into (4.126) gives

$$\begin{aligned}
\ddot{\mathbf{u}}(t+\Delta t) &= \frac{1}{\alpha}\left\{\frac{1}{(\Delta t)^2}\left(\mathbf{u}(t+\Delta t) - \mathbf{u}(t)\right) - \frac{1}{\Delta t}\dot{\mathbf{u}}(t) - (\tfrac{1}{2}-\alpha)\ddot{\mathbf{u}}(t)\right\} \\
\dot{\mathbf{u}}(t+\Delta t) &= \dot{\mathbf{u}}(t) + \frac{\delta}{\alpha}\left\{\frac{1}{(\Delta t)}\left(\mathbf{u}(t+\Delta t) - \mathbf{u}(t)\right) - \dot{\mathbf{u}}(t) - \Delta t(\tfrac{1}{2}-\alpha)\ddot{\mathbf{u}}(t)\right\} \\
&\qquad + \Delta t(1-\delta)\ddot{\mathbf{u}}(t)
\end{aligned} \tag{4.127}$$

Substituting (4.127a) into the FE equations (4.92) written at time $t+\Delta t$ then leads to

Newmark Scheme:

$$\left(\frac{1}{\alpha(\Delta t)^2}\mathbf{M} + \mathbf{K}\right)\mathbf{u}(t+\Delta t) = \mathbf{F}(t+\Delta t) + \mathbf{M}\left[\frac{1}{\alpha(\Delta t)^2}\mathbf{u}(t) + \frac{1}{\alpha\Delta t}\dot{\mathbf{u}}(t) + \frac{1}{\alpha}(\tfrac{1}{2}-\alpha)\ddot{\mathbf{u}}(t)\right] \tag{4.128}$$

The scheme is unconditionally stable for $\delta \geq 1/2, \alpha \geq (\delta + \tfrac{1}{2})^2/4$.

5. The Trapezoidal Scheme (Implicit)

This is the Newmark scheme with $\delta = 1/2, \alpha = 1/4$, which are the parameters which generally give the best accuracy. This is also called the constant-average-acceleration method because the expression for *u* becomes

$$u(t+\Delta t) = u(t) + \Delta t\dot{u}(t) + \frac{(\Delta t)^2}{2}\left(\frac{1}{2}\left(\ddot{u}(t+\Delta t) + \ddot{u}(t)\right)\right) \tag{4.129}$$

This is a Taylor series with, instead of the usual $\ddot{u}(t)$, the average over the interval, $\left(\ddot{u}(t+\Delta t)+\ddot{u}(t)\right)/2$.

4.2.4 Mode Superposition

As with the first-order system, the system of coupled ODEs $\mathbf{M}\ddot{\mathbf{u}}+\mathbf{K}\mathbf{u}=\mathbf{F}$ can be rewritten in terms of generalised coordinates $\mathbf{z}$ and $\dot{\mathbf{z}}$, so that the equations become uncoupled, and each can be solved independently of the others.

The analysis is essentially the same as for the first order system. Assuming that the eigenvalues and eigenvectors have been calculated, the eigenvectors are normalised through the equation

$$\overline{\mathbf{u}}^{(i)\mathrm{T}}\mathbf{M}\overline{\mathbf{u}}^{(j)}=\begin{cases}1, i=j\\ 0, i\neq j\end{cases} \tag{4.130}$$

for $i,j=1\cdots n$. Next, define the matrix $\mathbf{\Phi}$ whose columns are the eigenvectors $\overline{\mathbf{u}}^{(i)}$ and the diagonal matrix $\mathbf{\Omega}^2$ whose elements are the *n* eigenvalues:

$$\mathbf{\Phi}=\begin{bmatrix}\overline{u}_1^{(1)} & \overline{u}_1^{(2)} & \cdots & \overline{u}_1^{(n)}\\ \overline{u}_2^{(1)} & \overline{u}_2^{(2)} & \cdots & \overline{u}_2^{(n)}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{u}_n^{(1)} & \overline{u}_n^{(2)} & \cdots & \overline{u}_n^{(n)}\end{bmatrix},\qquad \mathbf{\Omega}^2=\begin{bmatrix}\omega^{(1)2} & 0 & \cdots & 0\\ 0 & \omega^{(2)2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \omega^{(n)2}\end{bmatrix} \tag{4.131}$$

The *n* solutions to the eigenvalue problem $\left[\mathbf{K}-\omega^{(i)2}\mathbf{M}\right]\overline{\mathbf{u}}^{(i)}=0$ can then be written in the form $\mathbf{K}\mathbf{\Phi}=\mathbf{M}\mathbf{\Phi}\mathbf{\Omega}^2$. With the eigenvectors **M**-orthonormalised as above, one has $\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}=\mathbf{I}$ and so, pre-multiplying the above equation by $\mathbf{\Phi}^{\mathrm{T}}$, $\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}=\mathbf{\Omega}^2$. Introduce next new generalised coordinates $\mathbf{z}$ and transform the original equations through $\mathbf{u}(t)=\mathbf{\Phi}\mathbf{z}(t)$. Thus, multiplying the equations by $\mathbf{\Phi}^{\mathrm{T}}$ gives

$$\dot{\mathbf{z}}+\mathbf{\Omega}^2\mathbf{z}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{F} \tag{4.132}$$

These equations are now uncoupled and each equation can be integrated in turn to evaluate the coordinates $z^{(i)}$, whence the nodal values of $u(t)$ can be evaluated, through $\mathbf{u}(t)=\mathbf{\Phi}\mathbf{z}(t)$. For this purpose one needs the initial conditions on $\mathbf{z}(t)$. Since $\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}=\mathbf{I}$, then $\mathbf{u}(t)=\mathbf{\Phi}\mathbf{z}(t)$ becomes $\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{u}(t)=\mathbf{z}(t)$ so that

$$\begin{aligned}\mathbf{z}(0) &= \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{u}(0)\\ \dot{\mathbf{z}}(0) &= \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\dot{\mathbf{u}}(0)\end{aligned} \tag{4.133}$$

4.2.5 Stability

Here, the stability of the explicit central difference scheme is examined. As with the explicit Euler analysis, it is only necessary to consider the homogeneous modal equation

$$\ddot{z} + \omega^2 z = 0 \tag{4.134}$$

This equation has already been examined in detail in section 4.2.3 above (see Eqn. 4.93 and Eqns. 4.99-104). That analysis now leads directly to the stability criterion $\Delta t \le 2/\omega$. The critical time step for stability will depend on the largest natural frequency in the system, giving the criterion (4.98).

4.3 Application: Elastodynamics

Here the problem of an elastic material subject to arbitrary loading and initial conditions is examined. This problem is examined in detail in Solid Mechanics, Part II, section 2.2, but the main points are discussed again here.

The geometry of the problem is as shown in Fig. 4.7, a (one dimensional) rod of length *l*, cross section A, subjected to a given displacement or stress/force at its ends. The Young's modulus of the rod is *E*.

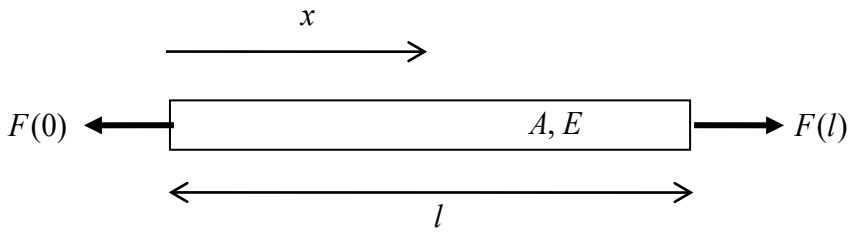

Figure 4.7: The Elastic Rod

4.3.1 Governing Differential Equation

The equations governing the response of the rod are:

Governing Equations for Elastodynamics:

Equation of Motion:

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{4.135}$$

Strain-Displacement Relation:

$$\varepsilon = \frac{du}{dx} \tag{4.136}$$

Constitutive Relation:

$$\sigma = E\varepsilon \tag{4.137}$$

The first two of these are derived in the Appendix to Chapter 2, §2.12.2 (where the body force is neglected[7]). The third is Hooke's law for elastic materials.

In these equations, σ is the stress, ε is the small strain (change in length per original length), u is the displacement and E is the Young's modulus of the material.

[7] the body force is usually much smaller than the other terms in dynamic problems

The strain-displacement relation and the constitutive equation can be substituted into the equation of equilibrium to obtain

1D Governing Equation for Dynamic Elasticity:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \qquad c = \sqrt{\frac{E}{\rho}} \tag{4.138}$$

This is the **one-dimensional wave equation,** and is the second order equation considered in Eqn. 4.76. The solution predicts that a wave emanates from a struck end of the rod with speed c. As the wave passes a certain point in the material, the material particles undergo a small displacement u and suffer a consequent stress [8].

Material	$\rho\ (\mathrm{kg/m^3})$	$E\ (\mathrm{GPa})$	$c\ (\mathrm{m/s})$
Aluminium Alloy	2700	70	5092
Brass	8300	95	3383
Copper	8500	114	3662
Lead	11300	17.5	1244
Steel	7800	210	5189
Glass	1870	55	5300
Granite	2700		3120
Limestone	2600		4920
Perspex			2260

Table 4.1: Elastic Wave Speeds for Several Materials

The stressed material undergoes longitudinal vibrations, with the particles oscillating about some equilibrium position. One should be clear about the distinction between the velocity of the oscillating particles, say $v = du/dt$, and the speed of the travelling stress wave, c.

For example, suppose that the bar is given a sudden displacement u_0 at time $t = 0$, Fig. 4.8.

[8] it was assumed that the density ρ in this analysis is constant. In fact, as the wave passes, the material gets compressed and the density of a constant mass of material increases. It can be shown that these fluctuations in density are, however, second-order effects and can be neglected

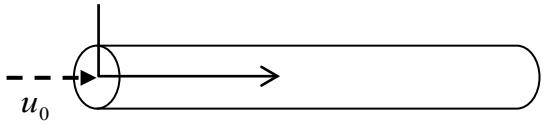

Figure 4.8: A sudden displacement prescribed at one end of an elastic rod

In that case the wave will travel from left to right, Fig. 4.9. As it passes a point, the material there will experience a sudden stress – the stress is discontinuous at the wave front. Eventually the wave will reach the other end of the bar and get reflected – there is then reinforcement and cancellation of waves as they meet each other in opposite directions.

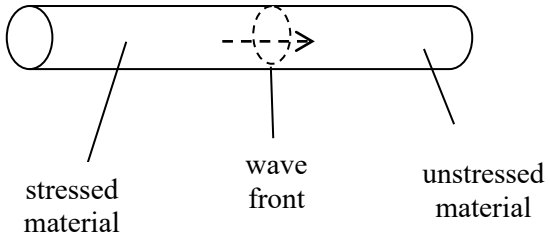

Figure 4.9: wave propagation along an elastic rod

Boundary and Initial Conditions

The case of a static rod was examined in §2.10. As in the static case, one must

$$\begin{aligned} &\text{specify } u(0,t) \quad \text{or} \quad \frac{\partial u}{\partial x}(0,t) && \text{B.C. at } x=0 \\ &\text{specify } u(L,t) \quad \text{or} \quad \frac{\partial u}{\partial x}(L,t) && \text{B.C. at } x=L \end{aligned} \tag{4.139}$$

Also, one must

$$\begin{aligned} &\text{Specify } u(x,0) && \text{I.C. for displacement} \\ &\text{Specify } \frac{\partial u}{\partial t}(x,0) && \text{I.C. for velocity} \end{aligned} \tag{4.140}$$

References to some exact solutions to the wave equation are given in the Appendix to this Chapter, as is a review of the dynamics of a single degree of freedom.

4.3.2 The FEM Solution

The only difference between this case and the static case is the inclusion of the acceleration term

$$\frac{\partial^2 u}{\partial t^2} \rightarrow \int_0^L \frac{\partial^2 u}{\partial t^2}\omega(x)dx \rightarrow \left[\int_0^L \omega_i \omega_j(x)dx\right]\ddot{u}_i \tag{4.141}$$

which leads to the mass matrix (the term inside the square brackets), so-called since the complete term, M times the nodal acceleration $\ddot{u}_i$ gives a force.

Eigenvalues (Natural Frequencies) and Eigenvectors (Mode Shapes)

The natural frequencies for the two-linear-element FE model of §4.2.2, are (as in Eqn. 4.87),

$$\omega^{(1)} = 1.61142\frac{c}{l}, \quad \omega^{(2)} = 5.62930\frac{c}{l} \tag{4.142}$$

The number of natural frequencies in a system will equal the number of degrees of freedom in the system. This compares with the real physical system, which has an infinite number of degrees of freedom and natural frequencies associated with the infinite number of material particles in the rod. To obtain a solution for the higher frequencies $\omega^{(3)}, \omega^{(4)}, \cdots$, it is necessary to include more degrees of freedom, i.e. elements, into the FE mesh. The FE solution for the natural frequencies, solved for 1, 2, 3 and 4 elements, is as tabulated below.

The exact solution for the frequencies is also tabulated; these are given by (see the Appendix for a reference to this)

$$\omega^{(n)} = \frac{(2n-1)\pi}{2}\frac{c}{l} \quad (n = 1,2,\ldots) \qquad = \frac{\pi}{2}\frac{c}{l}, \frac{3\pi}{2}\frac{c}{l}, \frac{5\pi}{2}\frac{c}{l}, \cdots \tag{4.143}$$

No. of elements	1	2	3	4	Exact
ω_1	1.7321 c/l	1.6114 c/l	1.5888 c/l	1.5809 c/l	1.5708 c/l
ω_2		5.6293 c/l	5.1962 c/l	4.9872 c/l	4.7124 c/l
ω_3			9.4266 c/l	9.0594 c/l	7.8539 c/l
ω_4				13.1007 c/l	10.9956 c/l
ω_5					14.1372 c/l
ω_6					17.279 c/l

Table 4.2: Natural Frequencies for 2-noded Linear Elements ($\mathbf{u} = \mathbf{0}$ at one end)

Note the following:

- the FE results for the *lower frequencies are more accurate* than those for the higher frequencies. This will be explained below.
- the FE model yields natural frequencies which are *higher than the true values*. This is because the FE model is a constrained version of the real system – it is not allowed the same degree of freedom as the real system – the FE model of the material is stiffer (see the case of a single degree of freedom, Eqn. 4.80, and the Appendix to this Chapter, §4.5.1, where $\omega = \sqrt{k/m}$, k being the stiffness).

Mode Shapes

Continuing again the above two element example, the eigenvectors or *modes* $\overline{\mathbf{u}}$ are now obtained from

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right)\overline{\mathbf{u}} = \mathbf{0}, \qquad \frac{2c^2}{l}\begin{bmatrix} 2-4\alpha & -1-\alpha \\ -1-\alpha & 1-2\alpha \end{bmatrix}\begin{bmatrix} \overline{u}_2 \\ \overline{u}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4.144}$$

one for each frequency:

$$\omega^{(1)}: \quad \overline{\mathbf{u}}^{(1)} = \begin{bmatrix} \overline{\mathbf{u}}_2 \\ \overline{\mathbf{u}}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \qquad \omega^{(2)}: \quad \overline{\mathbf{u}}^{(2)} = \begin{bmatrix} \overline{\mathbf{u}}_2 \\ \overline{\mathbf{u}}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} \tag{4.145}$$

The modes give the character of the system of equations, whose general solution is $\mathbf{u} = C_1\overline{\mathbf{u}}^{(1)}\sin\left(\omega^{(1)}t + \phi^{(1)}\right) + C_2\overline{\mathbf{u}}^{(2)}\sin\left(\omega^{(2)}t + \phi^{(2)}\right)$. The mode shapes for this particular problem are plotted below in Fig. 4.10 (solid lines).

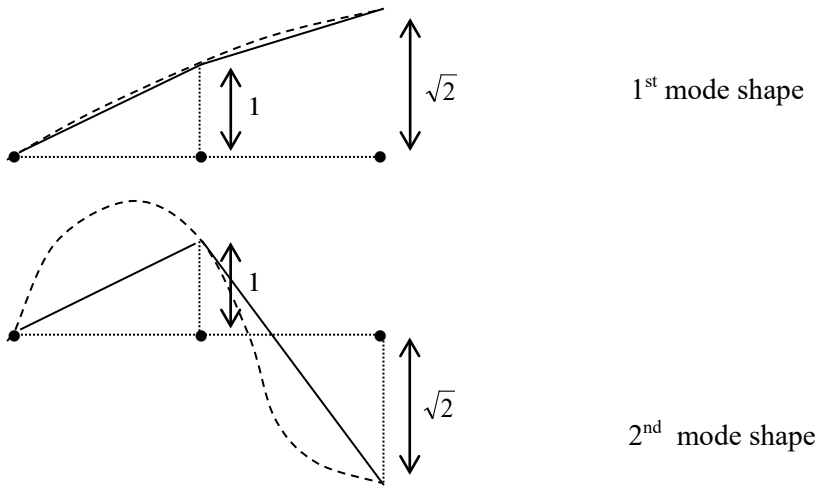

Figure 4.10: the first two mode shapes

These mode shapes can be compared to the exact shapes (see reference to these in the Appendix §4.5.2, and which are plotted in dotted lines:

$$\sin\left(\omega^{(n)}\frac{x}{c}\right)=\sin\left(\frac{(2n-1)\pi}{2}\frac{x}{l}\right)(n=1,2,\ldots)\quad=\sin\left(\frac{\pi}{2}\frac{x}{l}\right),\sin\left(\frac{3\pi}{2}\frac{x}{l}\right),\sin\left(\frac{5\pi}{2}\frac{x}{l}\right),\cdots \tag{4.146}$$

The simple linear two-element solution is not so bad an approximation for the first mode but, as with the natural frequencies, the higher, second, mode is not as well represented[9]. Note that the ratios of the amplitudes at the nodal points 2 and 3 are as the ratios of the exact solution.

The reason why the higher frequencies and corresponding mode shapes cannot be obtained with great accuracy is now clear. The higher modes contain many "waves" and one would need many elements to capture the features of this wave. These higher modes contain much more curvature than the lower modes and are difficult to model. For example, one would probably need five elements of equal length to capture the third mode with any real accuracy.

[9] the exact mode shapes here have been multiplied by $\sqrt{2}$ to fit the FE solution; the amplitudes of these shapes are unimportant as they depend on the initial conditions

Vibration Analysis

The above is a **vibration analysis**, where the natural frequencies and modes of the system are evaluated without regard to which of them might be important in an application and without regard to how the vibration is initiated. The exact combination of the modes for a particular problem is determined from the initial conditions (see below). The vibration is **free** if the "load vector" **F** is zero or constant (as in our case); **forced vibration** occurs when the load vector itself oscillates (is sinusoidal).

Complete Solution

Although the primary interest in this section was the determination and discussion of the frequencies and mode shapes, it is instructive to continue and solve the problem completely. The FE equations can only be solved exactly here because of the simplicity of this two-element problem.

To apply the initial conditions, it is best to rewrite the solution in the form

$$\mathbf{u} = \overline{\mathbf{u}}^{(1)}\left[A\cos\left(\omega^{(1)}t\right)+B\sin\left(\omega^{(1)}t\right)\right]+\overline{\mathbf{u}}^{(2)}\left[C\cos\left(\omega^{(2)}t\right)+D\sin\left(\omega^{(2)}t\right)\right] \tag{4.147}$$

Then, from $\mathbf{u}(0)=0$ and $\dot{\mathbf{u}}(0)=2x/l$, so that $\dot{u}_2(0)=1, \dot{u}_3(0)=2$, $A=C=0$,

$$\mathbf{u}(t)=\begin{bmatrix}u_2(t)\\u_3(t)\end{bmatrix}=\frac{1+\sqrt{2}}{2\omega^{(1)}}\begin{bmatrix}1\\\sqrt{2}\end{bmatrix}\sin\left(\omega^{(1)}t\right)+\frac{1-\sqrt{2}}{2\omega^{(2)}}\begin{bmatrix}1\\-\sqrt{2}\end{bmatrix}\sin\left(\omega^{(2)}t\right) \tag{4.148}$$

Using the shape functions, the complete solution is

$$u(x,t)=\frac{x}{l}\left\{\frac{1+\sqrt{2}}{\omega^{(1)}}\sin\left(\omega^{(1)}t\right)+\frac{1-\sqrt{2}}{\omega^{(2)}}\sin\left(\omega^{(2)}t\right)\right\} \qquad 1^{\text{st}} \text{ element}$$

$$u(x,t)=\frac{x/l+\sqrt{2}-1/\sqrt{2}}{\omega^{(1)}}\sin\left(\omega^{(1)}t\right)+\frac{x/l-\sqrt{2}+1/\sqrt{2}}{\omega^{(2)}}\sin\left(\omega^{(2)}t\right) \qquad 2^{\text{nd}} \text{ element} \tag{4.149}$$

with x here measured from the left-hand end. This can be compared to the exact solution (see reference to this in the Appendix to this Chapter),

$$u(x,t) = \frac{32l}{\pi^3 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}\sin(\lambda_n x)\sin(\lambda_n ct), \qquad \omega^{(n)} = \lambda_n c = \frac{(2n-1)\pi c}{2l}, \qquad n = 1,2,\ldots \tag{4.150}$$

The table below compares the FE and exact solutions (30 terms) for $x = l/4$ (in the first element) , with $l = 1, c = 5$ and $\Delta t = l/10c$. Considering the wave equation to represent the propagation of a wave through an elastic material at speed *c*, this time step Δt is one tenth the time a wave would take to travel the length of the bar.

	Δt	$2\Delta t$	$3\Delta t$	$4\Delta t$	$5\Delta t$	$6\Delta t$	$7\Delta t$	$8\Delta t$	$9\Delta t$	$10\Delta t$
Exact	0.310	0.620	0.930	1.240	1.550	1.860	2.170	2.465	2.651	2.715
FE	0.312	0.633	0.966	1.307	1.634	1.937	2.179	2.340	2.409	2.386

Table 4.3: Comparison of 2-element linear FE model with Exact Solution (for *u*(*l*/4))

Also shown, in Fig. 4.11, are the deformed shapes of the bar for the same 10 time intervals (using the exact solution, but plotted linearly through five points) – the 10th case is the maximum deformation, after which the displacement begins to decrease again, and then down to negative values.

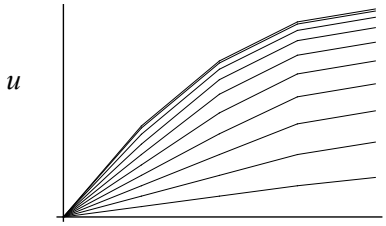

Figure 4.11: deformed shapes of the elastic bar (Note: this pricture gives the impression that the bar is swaying up and down (like a beam); actually, these displacements are along the direction of the rod – it is all one-dimensional)

Note the following:

- when a material is loaded or displaced, only a certain range of its natural frequencies are excited. For this reason it is not actually necessary to evaluate many of them in order to determine the material's response. When the loading itself is harmonic with frequency ω_f, then a general rule of thumb is that all the natural frequencies up to about $4\omega_f$ should be evaluated. By the same token, if the frequency of the loading function is very low, say one quarter of the lowest natural frequency or lower, then a *static* solution should yield an accurate result.

- in practice, in a model with hundreds or thousands of nodes, use of the standard method of solution for the eigenvalues and eigenvectors, expanding the determinant and solving the resulting polynomial, is not practical. Special techniques have been developed for this purpose (see advanced texts on FE and computational techniques).

Damping

The above solution for $u(x,t)$ continues to oscillate about $u = 0$ and does not decay with time. This is a characteristic of ideally elastic materials, for which there is no energy loss. In any real material, there will be *damping*, which dissipates energy and causes the amplitude of free vibration to decay with time[10]. A simple, somewhat artificial, way of introducing damping into the current model is to define a viscous damping matrix

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{4.151}$$

Here, α, β are constants to be determined experimentally (the former damps the lower modes whereas the latter has more of an effect on the higher modes). The FE equations are now

$$\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{F} \tag{4.152}$$

Note the following

- some FE software is capable of calculating damped natural frequencies. These frequencies are often only slightly smaller than the undamped natural frequencies (see Appendix 1 for the case of a single degree of freedom).
- in real transient problems, FE models will often incorporate some damping to eliminate resonance problems and oscillatory noise

Direct Integration

In most of the above, the free vibration model was analysed. The transient (or **dynamic**) response can be evaluated using one of the direct integration methods discussed earlier

In the implicit schemes, it is usual to take as the time-step the "element length" divided by the wave speed c, $\Delta t = L/c$, since this is the time taken for the wave to pass through the

[10] damping is a feature of many material models, for example of viscoelasticity and plasticity

element, but no further. Non-uniform and low- or high- order elements can be used, and when high-order elements are employed, a consistent mass matrix is usually appropriate.

Explicit schemes, with their smaller time-steps, are more appropriate for systems which are changing rapidly, for example for systems describing the sudden impact of materials. Implicit schemes are more appropriate for more slowly evolving systems, for example for systems describing the moderately paced flow of fluid through a porous medium.

Example

In the following graphs, Figs. 4.12-14, are plotted the solution to the wave equation with $u(0,t) = 0$, $u'(0,t) = F$. The explicit Euler scheme is used. The solution is compared with the "exact" modal equations solution (that is, obtaining a solution by first solving for the eigenvalues and eigenvectors, as done in the above).

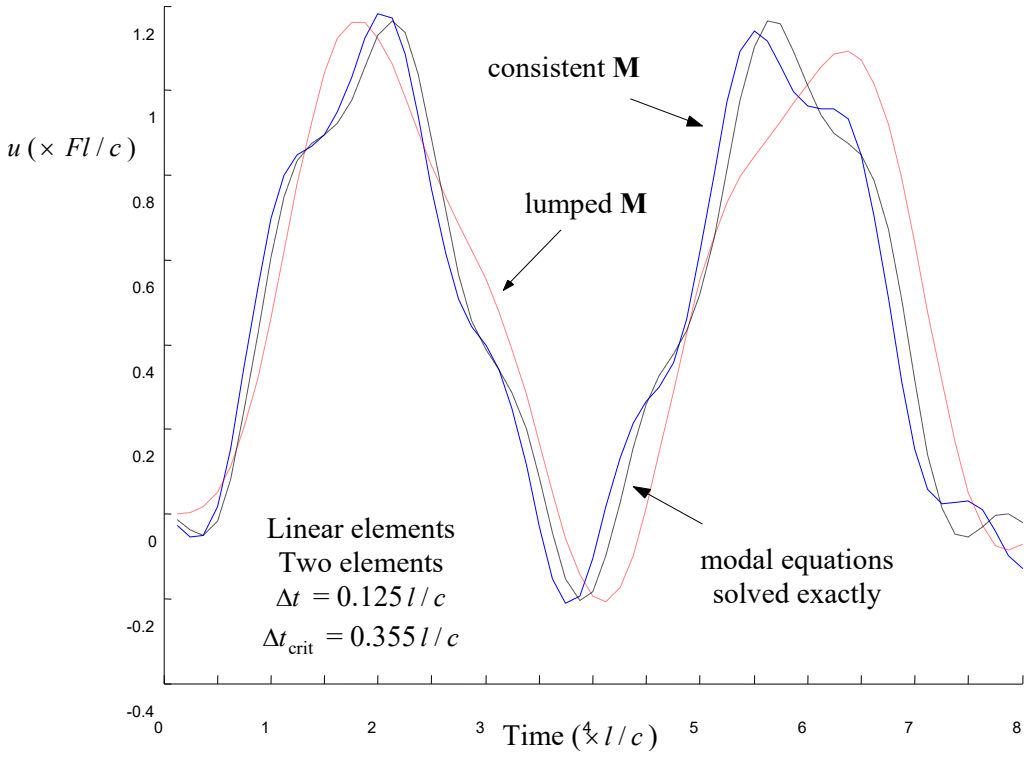

Figure 4.12: displacement at a material particle

unstable solution using the critical time step for two linear elements $\Delta t = 0.355\, l / c$

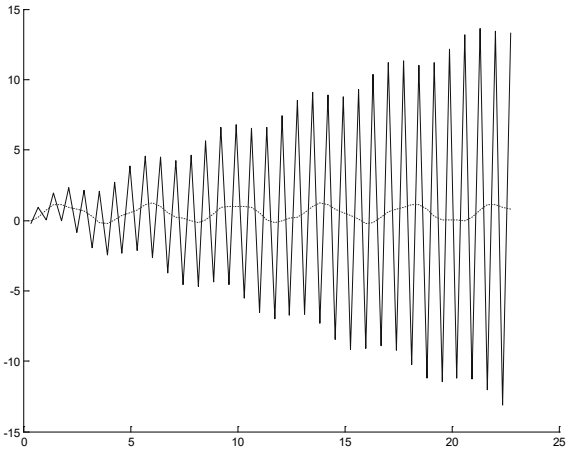

Figure 4.13: unstable displacement solution

solution using 20 linear elements

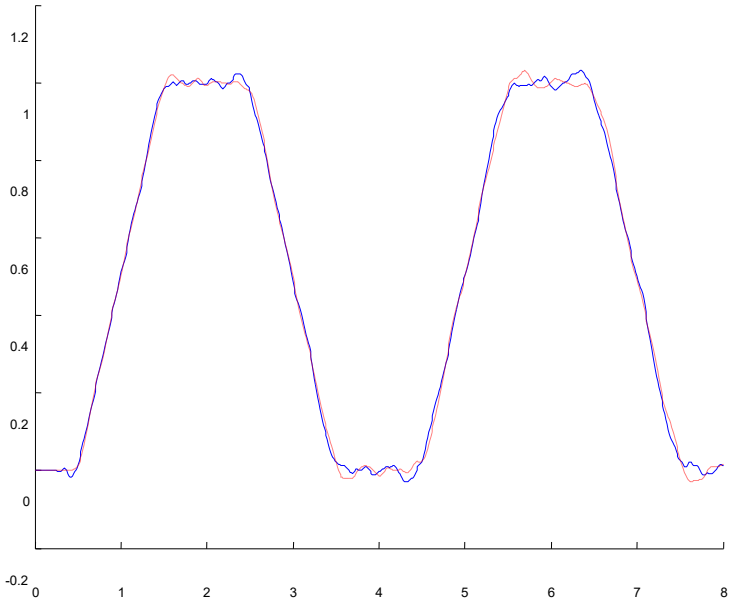

Figure 4.14: displacement at a material particle

Mode Superposition

As with the first-order system, the system of coupled ODEs (4.80) can be rewritten in terms of generalised coordinates $\mathbf{z}$ and $\dot{\mathbf{z}}$, so that the equations become uncoupled, and each can be solved independently of the others.

The mode superposition method is well-suited to problems which are dominated by the lower modes, and where the response of the higher modes is unimportant and can be neglected. This occurs for example with earthquake loading, where only the lowest 10 modes or so need to be considered, even though the order of the system may be quite large. On the other hand, for blast or shock loading, many more modes generally need to be included, perhaps about two-thirds of them. If this is the case then direct integration may be a more suitable solution procedure.

4.4 Problems

1. Derive the system of equations (4.11) for the 4-element model of the first order equation (4.5).
2. Consider the equation

$$q\frac{\partial^2 p}{\partial x^2} = \frac{\partial p}{\partial t}, \qquad 0 \le x \le l$$

   a) derive the **C** matrix (linear element)
   b) consider the boundary conditions $p(0) = 0$, $p(l) = A$. How many eigenvalues/modes would there be in a two-element FE model of this? Evaluate them.
   c) is it true that $\lambda_{\max} \propto 1/L^2$?
3. Use equations (4.42-44) to derive the Implicit-Euler algorithm (4.45).
4. Considering the Semi-Implict Algorithm for first order systems, show that the Crank-Nicholson scheme, $\alpha = 1/2$, leads to a truncation error proportional to $(\Delta t)^2$ [hint: expand $u(t_0 \pm \frac{1}{2}\Delta t)$ in Taylor series and subtract.]
5. Derive the relations (4.60) for the *C*-normalised eigenvectors (4.58).
6. What is matrix lumping? When and why is it done?
7. What is mode superposition and when might it be used to advantage?
8. Expand $\ddot{u}(t+\tau)$, $\ddot{u}(t+\Delta t)$ in Taylor series and hence derive the linear acceleration scheme formula (4.111) and deduce the truncation error involved.
9. Derive the Wilson θ equations (4.118).
10. When would you use an explicit scheme and when an implicit scheme? Why?
11. In a FE solution for the natural frequencies of the elastodynamic problem, which frequencies are more accurate? Why? What about the corresponding eigenvectors?

4.5 Appendix to Chapter 4

4.5.1 Review of the Dynamics of a Single Degree of Freedom

Free Vibration: No Damping

Consider a mass *m* attached to a freely oscillating spring, at initial position x_0 and with initial velocity $\dot{x}_0$. From Newton's Law

$$m\ddot{x} = -kx \tag{4A.1}$$

where *k* is the spring constant. This 2nd order ODE can solved to obtain

$$x(t) = x_0 \cos \omega t + (\dot{x}_0 / \omega) \sin \omega t \tag{4A.2}$$

where the frequency is $\omega = \sqrt{k/m}$ and the period of vibration is $T = 2\pi / \omega$. Different initial conditions simply shift the oscillations along the *t* axis, which can be seen by rewriting the displacement as

$$x(t) = A \sin(\omega t + \phi) \tag{4A.3}$$

where $\tan \phi = \omega x_0 / \dot{x}_0$, $A = \sqrt{x_0^2 + (\dot{x}_0^2 / \omega)^2}$. Shown in Fig. 4A.1 is a plot with $x_0 = 1, \dot{x}_0 = 7$ and $\omega = 3$ (so that there is one complete cycle every $T = 2\pi / 3 \approx 2.1\text{s}$.

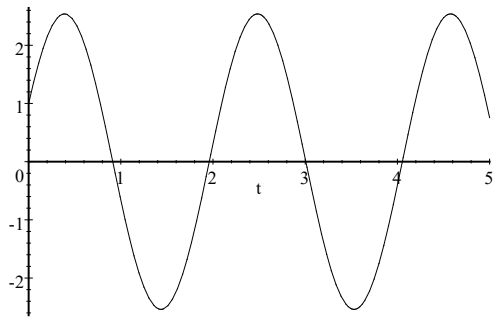

Figure 4A.1: Free Vibration

Consider now a constant force *P* applied to the oscillating mass, so that

$$m\ddot{x} = -kx + P \tag{4A.4}$$

This can be solved to obtain

$$x(t) = A\sin(\omega t + \phi) + \frac{P}{k} \tag{4A.5}$$

where $\tan\phi = \omega(x_0 - P/k)/\dot{x}_0$, $A = \sqrt{(x_0 - P/k)^2 + (\dot{x}_0^2/\omega)^2}$. It can be seen that the frequency is the same as in the unforced case and the mass oscillates about a mean position $x = P/k$, which is the static solution, that is, the position the mass would occupy if the force was applied very slowly and gradually from zero up to *P*.

Forced Vibration: No Damping

When an oscillatory force is applied, say $P = P_0\sin(\Omega t + \Phi)$, the solution to the non-homogeneous ODE is

$$x(t) = A\sin(\omega t + \phi) + \frac{P_0/k}{1 - (\Omega/\omega)^2}\sin(\Omega t + \Phi) \tag{4A.6}$$

and A, ϕ depend on the initial conditions (but are lengthy in this case). This is a superposition of two harmonic oscillations. Note that the amplitude becomes very large as $\Omega \to \omega$, a situation known as **resonance**.

Free Vibration: Damping

If one now also has a viscous damper with force $c\dot{x}$, then

$$m\ddot{x} = -c\dot{x} - kx \tag{4A.7}$$

When the damping is very large, $c^2 > 4mk$, the solution is of the form $x = Ae^{\beta_1 t} + Ae^{\beta_2 t}$ where $\beta_1, \beta_2 < 0$ and so the displacement falls quickly to the equilibrium position. If, on the other hand, the damping is not so high, then

$$x = e^{-\frac{c}{2m}t}\{A\cos(\omega_d t) + B\sin(\omega_d t)\}, \qquad \omega_d = \omega\sqrt{1-\xi^2}, \qquad \xi = \frac{c}{2m\omega} \tag{4A.8}$$

Here, ω is the undamped frequency and ξ is called the damping ratio.

Forced Vibration: Damping

Now we consider the system

$$m\ddot{x} = -c\dot{x} - kx + P_0 \sin(\Omega t + \Phi) \tag{4A.9}$$

The solution to the corresponding homogeneous equation is of the form $x(t) = \exp(-ct/2m)\{A\cos(\omega_d t) + B\sin(\omega_d t)\}$. This part of the solution dies away after a sufficient amount of time and is known as the **transient solution**. What remains is the particular solution,

$$x(t) = A\sin(\Omega t + \Phi - \alpha), \qquad \tan\alpha = \frac{2(\Omega/\omega)\xi}{1-(\Omega/\omega)^2} \tag{4A.10}$$

with

$$A = \frac{P_0/k}{\sqrt{\left(1-(\Omega/\omega)^2\right)^2 + \left(2(\Omega/\omega)\xi\right)^2}} \tag{4A.11}$$

4.5.2 Exact Solution to the 1-D Wave Equation

As mentioned above, the exact solution to the 1-D wave equation is detailed in Solid Mechanics, Part II, section 2.2 (see section 2.2.6).